MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXVI. MARCH 10, 1916. No. 5

Professor R. A. Sampson, M.A., D.Sc., F.R.S., PRESIDENT, in the Chair.

Joseph Mason Baldwin, M.A., D.Sc., Acting Government Astronomer for Victoria, Observatory House, South Yarra, Victoria, Australia;

Rev. Peter David Morse, L.T.S.C., Llain Cant, Wolf's Castle, S.O., Pembrokeshire; and

William Strachan, 30 Tregonwell Road, Bournemouth,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

John Botterill Rodney Haines, F.R. Met. Soc., 14 Chaucer Road, Wanstead, Essex; and H.M.S. *Umgeni*, c/o G.P.O., London (proposed by the Rev. John Spence);

Gertrude Longbottom, M.A., The Hollies, Louth, Lincolnshire (proposed by J. K. Fotheringham); and

Mrs. Proctor-Smyth, Lecturer on Astronomy, 9 Orchard Road, Altrincham, Cheshire (proposed by the Rev. A. L. Cortie).

Seventy-five presents were announced as having been received since the last ordinary meeting, including, amongst others:—

Scientific Papers of Sir G. H. Darwin, vol. v.; Supplementary Volume, containing biographical Memoirs, etc., presented by the Editors; Napier Tercentenary Memorial Volume, presented by the Royal Society of Edinburgh; and Ptolemy's Catalogue of Stars: a revision of the Almagest, by C. H. F. Peters and E. B. Knobel, presented by Mr. E. B. Knobel.

Statistics of the Minor Planets; with a Remark on the Orbital Planes of the Major Planets. By H. C. Plummer, M.A.

1. In a thesis by Elva G. Utzinger,* recently published, series of counts are given for the minor planets falling within certain ranges of the several orbital elements. The numbers are represented in tabular form and also in diagrams. They refer to 809 planets, and practically no attempt is made to smooth the irregularities which occur. The counts are thus useful raw material from which it remains to deduce the laws of distribution. It may be advantageous to find these laws even in a rough, empirical form, and an attempt is now made to do this.

2. The distribution of perihelia in longitude is taken first. As this is a circular distribution, harmonic analysis is obviously suggested. The first term has a considerable coefficient, and seems

TABLE I.

λ.	*n.*	*m.*	*n*−*m.*	λ.	*n.*	*m.*	*n*−*m.*
0°–10°	34	33	+1	180°–190°	12	11	+1
10–20	41	34	+7	190–200	18	11	+7
20–30	36	34	+2	200–210	10	11	−1
30–40	32	33	−1	210–220	14	12	+2
40–50	35	32	+3	220–230	18	13	+5
50–60	35	31	+4	230–240	14	14	0
60–70	35	30	+5	240–250	12	15	−3
70–80	21	28	−7	250–260	18	17	+1
80–90	21	26	−5	260–270	21	18	+3
90–100	25	25	0	270–280	16	20	−4
100–110	25	23	+2	280–290	18	22	−4
110–120	12	21	−9	290–300	19	24	−5
120–130	18	19	−1	300–310	23	26	−3
130–140	16	17	−1	310–320	23	28	−5
140–150	16	16	0	320–330	35	29	+6
150–160	15	14	+1	330–340	37	31	+6
160–170	14	13	+1	340–350	22	32	−10
170–180	11	12	−1	350–360	36	33	+3

sufficient by itself to represent the main variation. This gives for the number falling within an arc of 10°

$$m = 22{\cdot}4 + 11{\cdot}2 \sin(\lambda + 73^\circ{\cdot}3) \quad . \quad . \quad . \quad (1)$$

The numbers m resulting from this formula are given in Table I. and compared with the actual counts n. The differences $(n - m)$

* *Popular Astronomy*, xxiii., 1915 June-July.

may probably be regarded as entirely fortuitous. The maximum occurs at $\lambda = 16^\circ{\cdot}7$ or 4° distant from the perihelion of Jupiter. It may be said that the perihelion of a minor planet is three times more likely to fall within 10° of the perihelion of Jupiter than within a similar range from Jupiter's aphelion.

3. The distribution of the ascending nodes, which can be treated similarly, is exceedingly irregular. There is no simple, marked feature such as that shown by the perihelia. The formula derived by analysis is

$$m = 22{\cdot}4 + 4{\cdot}0 \sin(\lambda + 2^\circ{\cdot}5) + 3{\cdot}2 \sin(2\lambda + 99^\circ{\cdot}4) \quad . \quad (2)$$

The numbers m which follow from this are given in Table II. and compared with the counts n. The residuals are large, but further

TABLE II.

λ.	n.	m.	n−m.	λ.	n.	m.	n−m.
0°–10°	34	26	+ 8	180°–190°	30	25	+ 5
10–20	16	26	− 10	190–200	12	24	− 12
20–30	21	26	− 5	200–210	29	22	+ 7
30–40	32	25	+ 7	210–220	23	21	+ 2
40–50	33	25	+ 8	220–230	13	19	− 6
50–60	9	24	− 15	230–240	21	17	+ 4
60–70	26	24	+ 2	240–250	14	16	− 2
70–80	22	23	− 1	250–260	17	16	+ 1
80–90	30	23	+ 7	260–270	17	15	+ 2
90–100	27	23	+ 4	270–280	10	15	− 5
100–110	27	24	+ 3	280–290	19	16	+ 3
110–120	11	24	− 13	290–300	22	17	+ 5
120–130	26	25	+ 1	300–310	18	19	− 1
130–140	30	26	+ 4	310–320	16	20	− 4
140–150	31	26	+ 5	320–330	19	22	− 3
150–160	22	26	− 4	330–340	24	23	+ 1
160–170	20	26	− 6	340–350	23	24	− 1
170–180	33	26	+ 7	350–360	32	25	+ 7

analysis would be unprofitable. It is natural to look for the second harmonic, since this term makes no distinction between the ascending and descending nodes, and the distinction is in any case a rather artificial one. The distribution is examined because the figures are immediately available, but it is not to be supposed that the orbits of the minor planets are specially related in space to the Earth's orbit. Hence the positions of the nodes on the ecliptic are not in themselves of very great significance.

4. "Kirkwood speaks of two well-marked maxima, each extending about sixty degrees, from 310° to 10°, and from 120° to

180°. He mentions the non-coincidence of either of these maxima with the arc from 46° to 130°, which contains the ascending nodes of all the large planets. . . . We find . . . that the two maxima he mentions are no more marked than several others. In general, the one hemisphere from 0° to 180° might be considered as a maximum, and from 180° to 360° a minimum."

Maxima of such breadth as 60° can hardly be called well marked. But it is to be noticed that they persist when the number of planets has increased from 271 to 809, and that they are put in evidence by the process of smoothing now adopted. The first mentioned is advanced in longitude, but the second retains its position exactly. On the other hand, the position of maximum for the *ascending* node is decided by the first harmonic, and therefore falls at longitude $\lambda = 87°{\cdot}5$, which is in the middle of the arc containing the ascending nodes of the major planets. The two maxima are, however, barely separated in the hemisphere from 0° to 180°, where, apart from large irregularities, the systematic distribution is very uniform. The true minimum near longitude 270° is fairly well marked.

5. The maximum frequency of the line of nodes, when no distinction is drawn between the ascending and descending nodes, is decided by the second harmonic. It occurs at $\lambda = 175°{\cdot}3,\ 355°{\cdot}3$. As stated, the distribution of the nodal line in the ecliptic is scarcely in itself a point of interest. The distribution in the plane of Jupiter's orbit, or on the invariable plane, may be very different. The ecliptic nodes of Jupiter and Mars are in longitudes 99°·4 and 48°·8, but the ascending node of Mars on the plane of Jupiter's orbit is near longitude 4°. This is not far from the above position for the line of nodes of the minor planets. But the comparison is uncertain owing to the use of the ecliptic as the plane of reference.

The circumstances of discovery must be borne in mind. Minor planets are chiefly discovered at northern observatories, and therefore probably in the six months of the northern winter. Few will be discovered in the summer, owing to the short nights. Further, they are generally observed near opposition, and therefore, owing to the season, between longitudes 0° and 180°. Since search is naturally devoted mainly to the region of the ecliptic, the nodes of planets at the time of discovery are more likely to be in this than in the opposite region of the sky. At first sight these considerations seem to explain the facts. This is not so, because there is nothing in the circumstances of discovery to discriminate between ascending and descending nodes. Hence there must really be more ascending than descending nodes along the first half of the ecliptic. The explanation is not to be found in habits of observation, but in the distribution of the orbital planes relative to the ecliptic.

6. Although it is known that the eccentricities of the minor planets include notably high values, it might be supposed, *a priori*, that very small values would occur most frequently. This is not so. The graphical representation of the distribution shows, on the

contrary that small values are distinctly avoided. The frequency starts apparently from zero, mounts fairly rapidly to a maximum, and then decays more or less steadily. Hence the form of distribution naturally suggested is

$$dm = \mathrm{N}z^r \exp.(-k^2z^2)dz$$

where in this case z is e, the eccentricity. Let

$$\mathrm{N}_r = \mathrm{N}\int_0^\infty z^r \exp.(-k^2z^2)dz$$

$$= \frac{\mathrm{N}}{2k^{r+1}}\Gamma\left(\frac{r+1}{2}\right).$$

When r is an odd integer

$$\Gamma\left(\frac{r+1}{2}\right) = \left(\frac{r-1}{2}\right)!$$

and when r is an even integer

$$\Gamma\left(\frac{r+1}{2}\right) = \frac{r-1}{2}\cdot\frac{r-3}{2}\cdot\cdots\frac{1}{2}\sqrt{\pi}.$$

Hence

$$\frac{\mathrm{N}_0}{\mathrm{N}} = \frac{\sqrt{\pi}}{2k}, \quad \frac{\mathrm{N}_2}{\mathrm{N}} = \frac{\sqrt{\pi}}{4k^3}, \quad \frac{\mathrm{N}_4}{\mathrm{N}} = \frac{3\sqrt{\pi}}{8k^5}, \quad \frac{\mathrm{N}_6}{\mathrm{N}} = \frac{15\sqrt{\pi}}{16k^7},$$

and

$$\frac{\mathrm{N}_1}{\mathrm{N}} = \frac{1}{2k^2}, \quad \frac{\mathrm{N}_3}{\mathrm{N}} = \frac{1}{2k^4}, \quad \frac{\mathrm{N}_5}{\mathrm{N}} = \frac{1}{k^6}.$$

A table can now be formed by taking the logarithm of N_r and differencing, thus:

r.	$\log \mathrm{N}_r$.	$\Delta \log \mathrm{N}_r$.	$\Delta^2 \log \mathrm{N}_r$.
0	$\log \mathrm{N} - \log k - \log 2 + \frac{1}{2}\log\pi$		
1	$\log \mathrm{N} - 2\log k - \log 2$	$-\log k - \frac{1}{2}\log\pi$	
2	$\log \mathrm{N} - 3\log k - \log 4 + \frac{1}{2}\log\pi$	$-\log k - \log 2 + \frac{1}{2}\log\pi$	+·1961
3	$\log \mathrm{N} - 4\log k - \log 2$	$-\log k + \log 2 - \frac{1}{2}\log\pi$	+·1050
4	$\log \mathrm{N} - 5\log k + \log\frac{3}{8} + \frac{1}{2}\log\pi$	$-\log k + \log\frac{3}{4} + \frac{1}{2}\log\pi$	+·0711
5	$\log \mathrm{N} - 6\log k$	$-\log k - \log\frac{3}{8} - \frac{1}{2}\log\pi$	+·0538
6	$\log \mathrm{N} - 7\log k + \log\frac{15}{16} + \frac{1}{2}\log\pi$	$-\log k + \log\frac{15}{16} + \frac{1}{2}\log\pi$	+·0431

On the other hand, let n be the actual number counted within a given small range of z. If the quantities

$$\mathrm{M}_1 = \Sigma n, \quad \mathrm{M}_2 = \Sigma nz, \quad \mathrm{M}_3 = \Sigma nz^2, \ \ldots$$

are calculated, these are to be identified, if possible, with the successive integrals N_r. A simple method consists in comparing the second differences of $\log \mathrm{M}_r$ and $\log \mathrm{N}_r$.

7. In the case of the eccentricities of the minor planets

$$M_1 = 809, \quad M_2 = 119{\cdot}31, \quad M_3 = 22{\cdot}228, \quad M_4 = 4{\cdot}8822.$$

Hence the following table is formed:

r.	$\log M_r$.	Δ.	Δ^2.	$-\log k$.	$\log N$.
1	2·9079				4·7709
2	2·0766	$\bar{1}$·1687		$\bar{1}$·2211	4·7731
3	1·3469	$\bar{1}$·2703	·1016	$\bar{1}$·2179	4·7719
4	0·6886	$\bar{1}$·3417	·0714	$\bar{1}$·2180	4·7710
			Mean	$\bar{1}$·2190	4·7717

Here the second differences show that M_r is to be identified with N_r, and that the law of distribution may be represented by dN_1. The first differences then give three values of $\log k$, of which the mean is adopted. Hence four values of $\log N$ are derived, all satisfactorily accordant, and again the mean is taken. The total number is $N/2k^2$ or 810, in good agreement with the actual number, 809, which is to be distributed. The constants of the distribution have therefore been found, and the number of planets to be expected within a range of eccentricity de is therefore

$$dm = 59100\, e \,.\, \exp.(-36{\cdot}5\, e^2) \,.\, de \qquad . \qquad . \quad (3)$$

and the number in a finite range between e_1 and e_2 is

$$m = 810\{\exp.(-36{\cdot}5\, e_1{}^2) - \exp.(-36{\cdot}5\, e_2{}^2)\}$$
$$= 810(10^{-15{\cdot}85 e_1{}^2} - 10^{-15{\cdot}85 e_2{}^2}).$$

The numbers m derived from this latter formula are given in Table III., and compared with the counts n. The irregularities

TABLE III.

e.	n.	m.	$n-m$.	e.	n.	m.	$n-m$.
·00–·03	22	26	−4	·23–·25	52	35	+17
·03–·05	48	45	+3	·25–·27	24	26	− 2
·05–·07	63	62	+1	·27–·29	11	19	− 8
·07–·09	66	75	−9	·29–·31	11	13	− 2
·09–·11	78	82	−4	·31–·33	6	9	− 3
·11–·13	89	83	+6	·33–·35	8	6	+ 2
·13–·15	73	81	−8	·35–·37	1	4	− 3
·15–·17	80	74	+6	·37–·39	2	2	0
·17–·19	71	65	+6	·39–·43	2	2	0
·19–·21	53	55	−2	·43–·47	1	1	0
·21–·23	47	45	+2	over	1	0	+ 1

are of a kind which could scarcely be avoided by any practical frequency function. The formula allows for three planets with an

eccentricity less than ·01 : but no such planet has yet been found. It may seem strange that the frequency function runs to $e = \infty$. But this is no practical defect. On the contrary, were the number of planets trebled, there should be only one with an eccentricity exceeding ·47, and one has already been found. The frequency law seems sufficiently well determined to deduce from it the eccentricity of maximum frequency or the mode. This is $e = 1/k\sqrt{2}$ or ·117, a value exceeding the eccentricity of any major planet except Mercury.

8. At first sight the frequency distribution of the inclinations, graphically represented, resembles that of the eccentricities. When 1° is taken as the unit of i,

$$M_1 = 809, \quad M_2 = 7674, \quad M_3 = 100790, \quad M_4 = 1672500.$$

Thus the following table is derived :

r.	$\log M_r$.	Δ.	Δ^2.
1	2·9079		
2	3·8850	0·9771	
3	5·0034	1·1184	0·1413
4	6·2234	1·2200	0·1016

These second differences suggest that M_r is intermediate between N_r and N_{r-1} if a law of the same type is to hold. Hence the exponent r of i in the distribution of M_1 is a proper fraction. As the integrals connected with the frequency function are then less simple than when the exponent is an integer, other forms of function were tried. Such attempts met with indifferent success. Accordingly, an endeavour was made to represent the counts n between given limits of the inclination directly by a formula of the type

$$m = N i^r 10^{-ai^2}$$

or

$$\log m = \log N + r \log i - ai^2.$$

It was then found that an excellent representation was obtained by a particularly simple choice of constants, namely,

$$\log m = 2{\cdot}044 + \tfrac{1}{2} \log i - (i/20)^2 \quad . \quad . \quad . \quad (4)$$

The comparison of the numbers resulting from this formula with the counts is shown in Table IV. The class divisions chosen are, unfortunately, rather coarse. As in the case of the eccentricities, the infinite distribution, so far from indicating an excessive number of very large values, is actually in defect in this respect.

9. The breadth of a division being 4 units (4°), the frequency function is given by

$$dm = \tfrac{1}{4} N \sqrt{i} \, . \, 10^{-ai^2} di$$
$$= \tfrac{1}{4} N \sqrt{i} \, . \, \exp. \, (- k^2 i^2) di$$

TABLE IV.

i.	*n*.	*m*.	*n* − *m*.
0°–4°	151	153	− 2
4–8	221	220	+ 1
8–12	198	197	+ 1
12–16	135	134	+ 1
16–20	57	73	− 16
20–24	26	32	− 6
24–28	16	12	+ 4
28–32	2	3	− 1
over	3	1	+ 2

where $k = 0759$. The above representation is imperfect, inasmuch as a polygon has in effect been substituted for the frequency curve. The effect is probably not very serious. The total number distributed is

$$M_1 = \tfrac{1}{4}N\int_0^\infty \sqrt{i}\,.\,\exp.\,(-k^2i^2)di$$

$$= \frac{N}{8k^{1\cdot5}}\Gamma(\tfrac{3}{4}) = \frac{N}{6k^{1\cdot5}}\Gamma(1\cdot75).$$

Now Dale's Tables (1903) give $\log\Gamma(1\cdot75) = 9\cdot96335$, and hence $M_1 = 811$, in good agreement with the number of planets, 809. The excess of 16 under *m* in Table IV. is therefore mainly due to the treatment of the frequency curve as a polygon.

The inclination of greatest frequency, according to the above law of distribution, is $1/2k$ or $6°\cdot59$.

10. It has been remarked that the distribution of the ascending nodes on the plane of the ecliptic is not in itself a feature of very special interest. In order therefore to gain a clearer idea of the arrangement of the orbital planes, the distribution of their north poles will be considered. The material is taken from the *B.J.*, 1916, where the elements of 753 planets are given. The plane distribution of the points

$$x = i\cos\lambda/4°, \quad y = i\sin\lambda/4°, \quad \lambda = \Omega - 90°$$

is represented in Table V. This shows at once that, generally speaking, the distribution of the poles is very symmetrical and regular. There is a slightly greater extension approximately in the direction of the solstitial colure. Also the mean pole cannot be far removed from the pole of the ecliptic.

11. From the table it follows that

$$\Sigma n = 753, \qquad \Sigma nx = +160, \quad \Sigma ny = +48,$$

$$\Sigma nx^2 = 2502, \quad \Sigma ny^2 = 3384, \quad \Sigma nxy = +181.$$

TABLE V.

y \ x	−6	−5	−4	−3	−2	−1	0	+1	+2	+3	+4	+5	+6	
+9								1						1
+8				1										1
+7										2				2
+6							1	1		1				3
+5				2	2	1	2	1	1	2	2			13
+4				1	1	2	3	5	2	2	1	1		18
+3			2	1	5	8	13	6	5	4	1			45
+2			2	8	6	13	23	14	12	3	1		1	83
+1	1	1		7	12	25	25	31	11	6	3	3		125
0				2	13	26	61	40	18	15	5		1	181
−1		1	1	3	15	17	40	29	7	10	3		1	127
−2		2	1	4	6	16	25	10	12	5	1			82
−3			1	3	6	9	3	5	3	6			1	37
−4			1		3	4	6	1	1	2	1			19
−5				1	1	2	3	1	1	1				10
−6				2	1		2				1			6
	1	4	8	35	71	123	207	145	73	59	19	4	4	753

The position of the mean pole is therefore given by

$$\bar{x} = +0{\cdot}2125 = +0^\circ{\cdot}850$$
$$\bar{y} = +0{\cdot}0637 = +0^\circ{\cdot}255$$

and is thus at the distance $0^\circ{\cdot}887$ from the pole of the ecliptic and in longitude $16^\circ{\cdot}7$. When this point is taken as the origin,

$$\Sigma n x'^2 = 2468, \quad \Sigma n y'^2 = 3381, \quad \Sigma n x' y' = +171.$$

Hence the momental ellipse is

$$3381\, x^2 - 342\, xy + 2468\, y^2 = 1,$$

of which the major axis is the line

$$y = 5{\cdot}52\, x,$$

which is directed towards longitude 80°. Referred to this axis the ellipse becomes

$$2437\, \xi^2 + 3412\, \eta^2 = 1$$

or

$$(1{\cdot}80\, \xi)^2 + (2{\cdot}13\, \eta)^2 = 1$$

for the mean planet. The "mean square" deviations of a pole from the axes are thus 8°·5 and 7°·2, whence the mean deviation from the origin (or mean pole) is 11°·2.

12. It is interesting to compare this last result with that which is obtained from the frequency function previously found for the distribution of i. According to this, the square of the mean inclination is

$$\int_0^\infty i^{2\cdot5} \exp.(-k^2i^2)di \div \int_0^\infty i^{0\cdot5} \exp.(-k^2i^2)di$$
$$= \frac{\Gamma(1\cdot75)}{2k^{3\cdot5}} \div \frac{\Gamma(0\cdot75)}{2k^{1\cdot5}}$$
$$= 3/4k^2.$$

Hence the "mean square" inclination is $\sqrt{3}$ times the mode (6°·59)

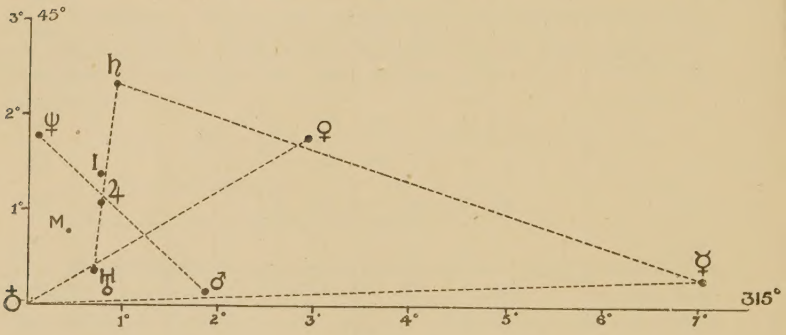


FIG. 1.

or 11°·4, in good agreement with the above deviation from the mean pole.

Here, to avoid misunderstanding, a remark may be made about this frequency function. It contains $\sqrt{i}$ as a factor, and therefore vanishes at the pole of the ecliptic. But it refers to the number of planets with inclinations between i and $i+di$, and the poles of the corresponding orbital planes cover an area proportional to $i\,di$. Hence the frequency function for a given small *area* contains $1/\sqrt{i}$ as a factor, and tends to infinity at the pole of the ecliptic. For this reason the avoidance of small inclinations is compatible with the great density of the poles in the neighbourhood of the pole of the ecliptic, which is apparent in Table V.

A really infinite distribution might have been secured by taking as the argument $\tan i$ instead of i, and in the case of the eccentricity $\tan \phi$ instead of e. But as the object is merely to represent the facts, and this has been done, no useful purpose would be served thereby.

13. Since the position of the mean pole of the orbital planes of the minor planets has now been determined, it seemed a matter of interest to compare this position with the corresponding poles

for the major planets. Accordingly, the poles have been plotted on a diagram (fig. 1) in which the point M represents the mean pole of the minor planets and I the pole of the invariable plane. Since the greatest extension is represented by the inclination of Mercury's orbit (7°), the diagram, which should be spherical, may be treated as plane.

A little consideration of this diagram immediately showed that it contained features of very great interest. A table of elements reveals the fact, doubtless familiar but yet remarkable, that the ascending nodes of Mars and Mercury differ in longitude by less than 2°, and the nodes of Venus and Uranus by less than 3°. Hence the poles Earth, Mars, Mercury, and the poles Earth,

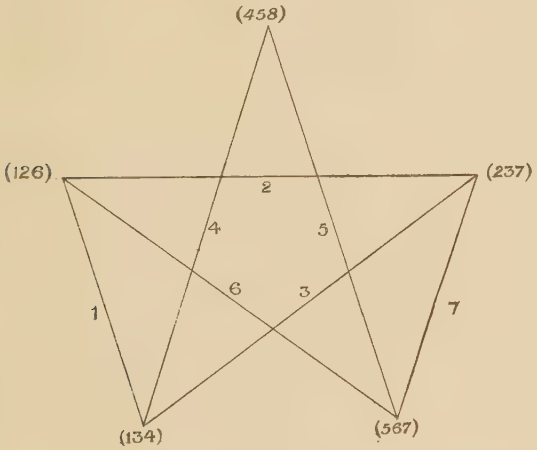


FIG. 2.

Venus, Uranus, are nearly collinear. But the diagram shows much more. It is at once apparent that the line Mercury–Saturn nearly passes through the pole of Venus. Then the line Saturn–Jupiter-Uranus is noticed, and finally a fifth line—Mars–Jupiter-Neptune. Hence the configuration of the planes of the major planets is subject to three approximate, empirical laws of a very curious kind:

(i.) The poles of the planes lie three by three on five lines. Otherwise, the orbital planes are concurrent, three by three.

(ii.) The pole of each orbit, with the exception of Neptune, lies on two of these five lines.

(iii.) Each line contains the orbital poles of two adjacent major planets.

14. The association of the third planet with each line is at first sight arbitrary. Yet it is also governed by a curious law. A star pentagon is shown (fig. 2) with two outer sides added by joining two pairs of corners. The sides of the inner pentagon are numbered alternately clockwise from 2 to 6; otherwise expressed,

the sides of the star pentagon are numbered consecutively. The numbers 1 and 7 are placed on the additional sides to the left and right respectively. Now, let 1 represent Mercury, 2 Venus, and so on in order up to 8 Neptune. Then the figure, which is very easily remembered, serves to indicate which sets of orbital planes are mutually and approximately concurrent. The orbit 8 is not represented by a side of the figure. But it passes through the intersection of 4 and 5, and is not otherwise conditioned. This exceptional position of Neptune seems appropriate to the outside planet of the system.

Neptune also presents an exception to the converse of (iii.). Otherwise every consecutive pair of poles determines a line, thus:

$$12-6 \qquad 45-8$$
$$23-7 \qquad \left\{\begin{matrix}56-7\\67-5\end{matrix}\right.$$
$$34-1$$
$$78-?$$

The vacant place in the last set might possibly be filled (fig. 1) by the mean minor planet. But the highly scattered poles of the minor planets are better left out of consideration in this connection. An arithmetical counterpart to the geometrical rule of association has not presented itself, though it is noticed that the sum of the third numbers on any one of the three lines above $= 6$ (mod. 8).

15. It seems scarcely credible that this beautiful law in the configuration of the planetary system can have escaped notice, but no reference to it is known. It requires only the evidence of a small scale diagram. The lines are drawn through the extreme poles, with no attempt at accommodation. Yet the greatest distance of an intermediate pole from the corresponding line may be estimated at 5′. Such discrepancies are probably within the limit of periodic perturbations, which hinder the law from being accurately observed at any particular time. Whether the configuration, assumed to exist, is stable is a problem of great complexity and probably of great difficulty. It is difficult to believe, however, that it is a chance arrangement which merely happens to be approximately true at the present time. The eight poles require sixteen co-ordinates to determine their positions. This number is reduced to eleven by the five independent relations subsisting. At the least these relations are certainly far more accurate than Bode's law for the mean distances. A closer analogy may perhaps be found in Cassini's laws for the motion of the Moon, which have been proved to correspond with a stable arrangement.

It may be noticed that the pole of the solar rotation, according to Carrington at 7° 15′ from the pole of the ecliptic in longitude 343° 40, is situated on the line Earth–Uranus–Venus. The orbits of these three planets intersect the plane of the Sun's equator very nearly in the same point. There is perhaps a certain tendency for the orbital poles of the minor planets to fall in straight lines. Such an appearance in a fairly dense distribution

is apt to be deceptive. But the counts reproduced in Table II. show very pronounced differences in the number of nodes falling in adjacent sectors of 10°, and this does suggest that the poles are distributed radially to some extent.

16. The results of this paper may be reviewed in a brief summary. It has been found that the distribution in longitude of the perihelia of the minor planets can be represented by one harmonic term (1). The maximum density is three times the minimum density, and occurs in longitude 16°·7, or 4° from the perihelion of Jupiter. This tendency to a similar orientation has long been noted, and was explained by Newcomb in 1862 as a result of secular perturbations by Jupiter.* Callandreau † pointed out the same tendency in the orbits of comets of short period (§ 2).

A frequency function (3) has been found which expresses the distribution of the minor planets in respect to the eccentricities of their orbits. According to this expression the most frequent eccentricity is in the neighbourhood of 0·117. Very small eccentricities are rare. It is well known that the problem of finding a preliminary circular orbit is not always soluble. In such cases, and when a minor planet has been insufficiently observed to allow of the elements being determined, it may be better to assume the eccentricity 0·12 and to make the position of perihelion approximately the same as that of Jupiter. In any case, a circular orbit is certain to be wide of the truth (§§ 6, 7).

The distribution of the planes of the minor planets has been studied by means of the orbital poles The mean centre is situated at 53′ from the pole of the ecliptic in longitude 16°·7. This distance is small in relation to the scattering. There is some evidence of greater extension of the poles in the direction of longitude 80°, the ratio of the axes being 7 : 6. The distribution about the mean pole is sensibly regular, and the standard deviation from the mean pole is 11°·2 (§§ 10, 11).

Since the mean pole is relatively close to the pole of the ecliptic, the frequency in respect to the inclinations of the orbits and the distribution of the ascending nodes in longitude are of more direct interest than they would otherwise be. A frequency function is found for the inclinations, the logarithm of which is expressed by (4). According to this formula, inclinations in the neighbourhood of 6°·6 are most frequent, and the standard (mean square) inclination is 11°·4, in agreement with the above independent determination (§§ 8, 9, 12).

The distribution of the ascending nodes is irregular. It is represented by two harmonic terms (2) of small amplitude. According to this mode of smoothing, the density is almost uniform from 0° to 180°, and there is a minimum at 270°. This is doubtless due to the position of the mean pole in longitude 16°·7 relatively to the pole of the ecliptic. The maximum of the second harmonic is at 175°·3, 355°·3. This may be connected with the greater scattering of the poles in the direction of longi-

* *Cf.* Callandreau, *Bull. Astr.*, xx. p. 411. † *Ibid.*

tudes 80°, 260°. The extreme irregularity of the distribution of the nodes may be more than accidental (§§ 3–5).

A diagram is given showing the positions of the orbital poles of the major planets and the mean pole of the minor planets relatively to the pole of the ecliptic. From this it appears that, within the limits of periodic perturbations, the planes of the major planets intersect three by three in five lines. A curious rule is found to connect the planets which are thus associated (§§ 13–15).

Dunsink Observatory:
1916 *February* 23.

A Note on Bright Meteors. By H. C. Plummer, M.A.

1. From a list of the real paths of meteors, such as have been published by Mr. Denning, a number of interesting conclusions can be drawn. Thus Mr. Denning refers to the relative heights at which different streams make their appearance. Here it is proposed to examine a few points of a general kind, in which no distinction is drawn between meteors of different families. The most recent list,* covering the years 1912 to 1915, will be referred to as List C. The preceding list,† covering the years 1897 to 1911, will be referred to as List B.

2. The distribution of the meteors in List C as regards velocity and height of appearance in the atmosphere is shown in Table I.

TABLE I.

Velocities (miles per hour).

Heights (miles)	0	10	20	30	40	50	60	70	80	90	100	
20		2	1									3
30												0
40		1	4	2	3							10
50			20	14	3	2	1					40
60			21	32	13	5	5					76
70			13	16	16	12	5	3				65
80			3	4	13	9	2	4		1	1	37
90				1	1	1		1			1	5
100		3	62	69	49	29	13	8	0	1	2	236

The velocities (V) and the heights (H) being expressed in whole

* *M.N.*, lxxvi. p. 219. † *M.N.*, lxxii. p. 423.

numbers, the limits of each class should be ½ less than the numbers shown in the margins. From this table we find:

Mean H = 68·0 miles.
Mean V = 30·0 miles per sec.
$\sigma(H) = 12{\cdot}3$ (standard deviation).
$\sigma(V) = 15{\cdot}2$ (standard deviation).
$r = {\cdot}497 \pm {\cdot}03$ (correlation coefficient).

3. The similar treatment of List B yields the following results:—

Mean H = 69·8 miles.
Mean V = 27·2 miles per sec.
$\sigma(H) = 12{\cdot}4$.
$\sigma(V) = 13{\cdot}0$.
$r = {\cdot}447 \pm {\cdot}03$.

Here the number of meteors for which the velocities are given is 317, or about three-quarters of the total. Five exceptional cases have been excluded by stopping at the limits H = 120, V = 80. But for one-quarter of the list no velocities are given, and for many of these meteors the description "swift" or "very swift" is substituted for a numerical estimate. It is to be inferred that the table drawn from List B is defective in the higher velocities. Hence it did not seem advisable to combine it with Table I., which is comparatively free from this defect. Under these circumstances, the agreement between the above sets of results seems very satisfactory. The lower values of the mean V, of $\sigma(V)$, and of the correlation in the second case are all attributable to the absence of the swifter class of meteors in the material discussed.

4. For this reason the results derived from List C, in spite of the smaller amount of material, are in some respects to be preferred. The mean height at which a meteor becomes visible may be taken to be 69 miles, or 111 km. The mean velocity of a bright meteor within the atmosphere is about 30 miles a second, or 48 km./sec. Between H and V the dependence is measured by a correlation coefficient 0·5. This figure is quite considerable, and accords with the expectation that a meteor will become luminous at a height depending on its speed, but also involving other factors such as the angle of incidence. When V is given

$$H = 0{\cdot}40V + 57,$$

but when H is given

$$V = 0{\cdot}62H - 13,$$

the choice between these two equations depending on whether the mean H or the mean V is required. A single equation is possible only when the correlation is complete. If it may be supposed that the determination of the time of flight presents the greater uncertainty, it is presumably the latter equation which is the more informative. This relation gives in numbers:

H	25	35	45	55	65	75	85	95	miles.
V	2	9	15	21	27	33	40	46	miles/sec.

Some deduction should be possible respecting the density of the atmosphere at different heights.

5. When it is a question of the distribution of the meteors in relation to the height at which they appear, the data from Lists B and C may be combined. Thus Table II. is formed. The numbers under B and C give the numbers of meteors in Lists B and C corresponding to the given ranges of H. It will be noticed that List B is relatively deficient in meteors appearing at a low altitude, and relatively excessive in meteors appearing at a high altitude. On the whole the distribution about the mean, which is taken at 69·5, appears to follow the normal Gaussian law. The numbers N represent

Table II.

H.	B.	C.	B+C.	B+C−N.	N.	H.	B.	C.	B+C.	B+C−N.
65–69	73	51	124	+21	103	70–74	53	33	86	−17
60–64	55	28	83	− 6	89	75–79	52	36	88	− 1
55–59	47	20	67	+ 1	66	80–84	33	23	56	−10
50–54	18	25	43	+ 1	42	85–89	40	16	56	+14
45–49	8	9	17	− 6	23	90–94	16	4	20	− 3
40–44	3	2	5	− 6	11	95–99	12	1	13	+ 2
35–39	1	...	1	− 3	4	100–104	5	...	5	+ 1
30–34	1	...	1	− 1	2	105–109	1	...	1	− 1
25–29	...	2	2	+ 1	1	110–114	4	...	4	+ 3
20–24	..	1	1	+ 1	...	115–119	...	...	...	0
					...	125–144	6			

$$N = 340 \int_{t_1}^{t_2} e^{-t^2} dt$$

where

$$t = 0{\cdot}055\ (H - 69{\cdot}5).$$

There may be some degree of skewness, but it is obscured by the natural errors of the samples. The distribution seems more nearly symmetrical than might have been expected from the conditions. The "probable error" is about 8·7 miles, so that half the meteors observed become visible at heights lying between 61 and 78 miles (98 and 126 km.). For the group of six meteors in List B, appearing at a height between 125 and 145 miles, the statistical distribution makes no allowance. It thus corroborates the scepticism expressed by Mr. Denning as to the heights much exceeding 100 miles.

6. It is of interest to notice the results* of Professor Störmer's observations of the height of the aurora in 1913. According to these the aurora begins at the level 90 km. (56 miles). The altitude of greatest density (or frequency) is about 106 km. (66

* *Nature*, vol. xcvii. p. 5.

miles). The aurora certainly seems to extend above 200 km. (124 miles), and a few isolated points have been observed above 300 km. These are possibly open to doubt. Here are two questions deserving close attention: the limiting altitude to which an aurora can extend, and the maximum height at which the atmosphere retains sufficient density to render a meteor luminous. Both require for their definite solution prolonged and skilful observations.

7. The distribution of the velocities must be deduced from List C, because the higher velocities are not recorded quantitatively in List B. The most striking feature is the relative absence of low velocities. It may be suspected that this is due not so much to the real absence of slow meteors as to the fact that very slow meteors are inconspicuous, and therefore escape observation in combined work such as provides the material for these lists. The present object, however, is to represent the observed distribution. With the unit 1 mile/sec. it is found that

$$M_1 = \Sigma n = 236, \qquad M_2 = \Sigma nV = 7235,$$
$$M_3 = \Sigma nV^2 = 273250, \qquad M_4 = \Sigma nV^3 = 1250800,$$

whence, in tabular form,

r.	$\log M_r$.	Δ.	Δ^2.	$-\log k$.	$\log N$.
1	2·3729	...	...	...	$\bar{2}$·0679
2	3·8594	1·4865	...	·8844	·0707
3	5·4366	1·5772	·0907	·8782	·0672
4	7·0972	1·6606	·0834	·8824	·0680
				(Means) ·8817	$\bar{2}$·0685

A very simple form of $\frac{n}{N}$ distribution which seems to suit this case fairly well is

$$dn = Nx^s e^{-kx} dx.$$

Then, if

$$N_s = N\int_0^\infty x^s e^{-kx} dx = \frac{N.s!}{k^{s+1}}$$

$$\Delta \log N_s = -\log k + \log s$$
$$\Delta^2 \log N_s = \log s - \log (s-1).$$

The above tabular second differences show that if this form is to be adopted, $s = r + 2$. Hence the values of $\log k$ and $\log N$ are derived, and the means are adopted.

8. The distribution of the observed velocities (in the unit mile/sec.) is therefore given by the formula

$$dn = [\bar{2}\cdot0685]V^3 e^{-0\cdot131V} dV$$
$$= [\bar{2}\cdot0685]V^3 10^{-0\cdot057V} dV,$$

and the number of velocities in a finite interval can be derived from

$$M_1 - m = N\int_V^\infty V^3 e^{-kV} dV$$
$$= Nk^{-4}(k^3V^3 + 3k^2V^2 + 6kV + 6)e^{-kV},$$

where $Nk^{-4} = 39{\cdot}4$. The differential form is generally sufficient. The comparison between the observed numbers (n) and the calculated numbers (n') is shown in Table III. (first four columns) :—

TABLE III.

	List C.			List B.		
V.	n.	n'.	$n-n'$.	n.	n''.	$n-n''$.
0–9	3	9	− 6	14	12	+ 2
10–14	19	20	− 1	34	27	+ 7
15–19	43	31	+12	54	41	+13
20–24	38	34	+ 4	53	45	+ 8
25–29	31	33	− 2	37	44	− 7
30–34	23	28	− 5	37	37	0
35–39	26	22	+ 4	33	29	+ 4
40–44	17	17	0	25	23	+ 2
45–49	12	12	0	15	16	− 1
50–54	5	8	− 3	10	19	− 9
55–59	8	6	+ 2			
60–64	6	4	+ 2	3	8	− 5
65–69	2	2	0			
over 70	3	4	− 1	2	5	− 3

9. According to the formula the mode, or most frequent velocity, is $3/k$ or 22·8 miles a second. The counts suggest that this is excessive. But the group 15–19 seems accidentally large. An empirical rule gives

$$\text{Mode} - \text{median} = 2\ (\text{median} - \text{mean}).$$

Now the mean is 30 and the median about 27, so that by this rule the mode is about 21, again outside the limits set by mere inspection.

The mean-square velocity by the formula is

$$\sqrt{(M_3/M_1)} = \sqrt{(N_5/N_3)} = \sqrt{20/k} = 34{\cdot}1.$$

The corresponding value from the mean and the standard deviation (§ 2) is 33·6. These numbers are equivalent to 55 and 54 km./sec. Now if V_0 is the circular orbital velocity of the Earth, $V_0\sqrt{2}$ is the parabolic velocity at the same distance from the Sun, and $V_0\sqrt{3}$ is the mean-square value of the relative velocity. This is 52 km./sec. Thus the numbers found can be fairly explained as the resultant of the Earth's velocity with an external meteoric velocity which is of the order of the parabolic orbital speed. In view of the disturbing factors which operate when a meteor enters the sphere of the Earth, and in the atmosphere, the agreement is as close as can be expected. But the excess actually found could be explained by a natural failure to record the slower, fainter meteors.

10. Though the material of List B seemed unsuitable as a foundation for the law of distribution in regard to speed, it is pos-

sible to compare the counts from this list with the law otherwise determined. This is done in the last three columns of Table III. Here n'' is simply $4n'/3$. As was to be expected, there is a relative excess among the smaller velocities and a relative defect among the larger. Again, the mode according to the theoretical distribution appears distinctly too great. The law, of course, is nothing more than a rough approximation. But though a better form might doubtless be found even from the present material, a really satisfactory result will require the accumulation of many more observations, so that the accidental deviations in the several groups may have a less disturbing effect.

The purpose of this note will be attained if it serves in some slight measure to reinforce Mr. Denning's plea for more efficient co-operation.

Dunsink Observatory:
1916 *March* 8.

The Eclipsing Binary TT Aurigæ. By C. Martin and H. C. Plummer. (Plate 5.)

1. The star TT Aurigæ (B.D. $+39^{\circ}\cdot1192$), the variability of which was discovered by Miss Leavitt* from an inspection of Harvard photographs, belongs to the β Lyræ type. This character was first definitely established by J. Balanowsky,† who made a careful photometric study of the light curve and determined the elements of the system. During the past two years Mr. Martin has secured 83 exposures on the region with the 15-inch reflector, and these serve to determine with some accuracy the photographic light curve. All measures and reductions have been made by him according to the method ‡ adopted at this observatory.

2. The magnitudes of the comparison stars were based on two plates exposed as nearly as possible under similar conditions, (*a*) on the region of the variable, and (*b*) on the χ Persei cluster. The magnitude scale of these plates is deduced by adopting Schwarzschild's magnitudes of certain standard stars in the latter region. Table I. gives the results.

TABLE I.

Star.	B.D.+39°.	Pl. 1427. m_1.	Pl. 1441. m_2.	Mean. m.	Corr.	Final. m'.	P.E.	Range.
a	1191	7·82	7·76	7·79	+·004	7·79	±·020	·16
b	1176	8·08	8·05	8·06	+·008	8·07	±·022	·18
c	1183	8·15	8·14	8·15	−·010	8·14	±·019	·16
d	1198	8·60	8·53	8·56	+·004	8·56	±·018	·18
e	1195	9·41	9·37	9·39	+·001	9·39	±·022	·13

* *Harvard Circular*, 130. † *Pulkowo Mitteilungen*, 57.
‡ *M.N.*, lxxiii. pp. 166, 440.

Here m_1 and m_2 are the magnitudes of the comparison stars deduced from the two plates, and m is the corresponding mean. This is adopted in the reduction of the exposures on the variable. A series of residuals is thus obtained for each star, the mean of which is given in the sixth column. By applying these corrections to m, the final set of photographic magnitudes m' is obtained. The corrections are, however, so small that it is unnecessary to revise the magnitudes of the variable, which are based on the system m. In the eighth column are given the probable errors for a single exposure which result from the series of residuals. The extreme range shown by the residuals appears in the last column. It should be added that the comparison star *e* was only added in October 1915, and used in the reduction of the last thirty plates.

3. According to the V.J.S. list, Enebo has given the elements for primary minimum

J.D. 241 8282·29 + 1·332705 E.

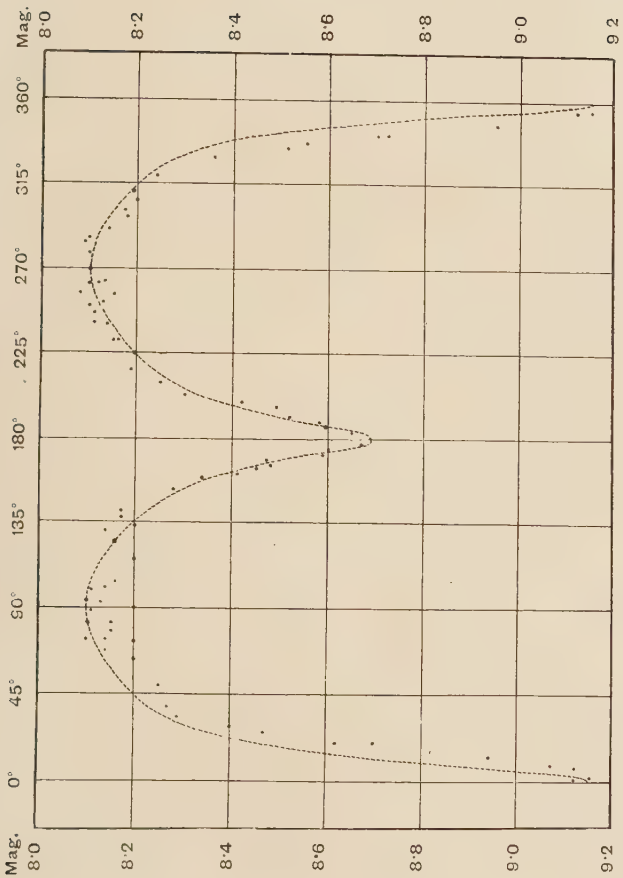
From the Dunsink observations the date deduced for minimum is

J.D. 242 0806·448.

In the interval 1894 periods have elapsed, and the Dunsink epoch is $0^d{\cdot}015$ later than the above elements indicate. Hence the period required to reconcile the two epochs is $1^d{\cdot}332713$, which has been used in the reductions. There is no consistent evidence of any change in the period.

4. The photographic magnitudes of TT Aurigæ are plotted on Plate 5. Here also is given for comparison the visual light curve based on the elements of the system which were derived by Balanowsky from his photometric observations. This curve is drawn with the help of the calculated magnitudes given by him in Tafel ii. (p. 127). All the change that has been made consists in adding 5^m to the times in the second column of this table in order to make the primary minima coincide better, and in adopting 7·74 as the magnitude of Balanowsky's comparison star *a*. By an obvious misprint this star is designated B.D. + 39°·1911. Presumably it is identical with the star *a* of the above Table I., according to which the photographic magnitude is 7·79, while the photometric magnitude (Potsdam system) is 8·36. Since the magnitudes of the variable are in this way well represented at maximum and at both minima, the conclusion is suggested that the colour index of both components is the same and sensibly equal to the colour index of the star *a*. All three stars are presumably of the same type, and for each the photometric magnitude exceeds the photographic by 0·6. This common colour index of −0·6 suggests that the type may be earlier than A, which has been assigned to TT Aurigæ.

5. Apart from the ranges at the critical phases, it will be seen that the photometric curve satisfies the photographic observations excellently, except during the six hours covering the primary



Photographic Magnitudes and Photometric Light Curve of TT Aurigæ.

eclipse. In this interval the systematic discrepancy is clear. The residuals given in Table II. lead to the following results:—

Hours.	No. of Obsns.	Mean Discordance (O − C).	Average Deviation.
		m	m
4–28	62	+ ·006	± ·026
0–4, 28–32	21	+ ·143	...

Thus if the curve could be accepted over the secondary minimum, including the two maxima, the probable error of a photographic magnitude of the variable would appear to be of exactly the same order as the probable errors for the comparison stars shown in Table I. This is appreciably better than the accuracy claimed ($\pm$ m·03) for a single exposure in previous work, where the standard of comparison was merely a curve derived by harmonic analysis from the observations themselves. So far as it goes, this is eminently satisfactory as evidence bearing on the variable star work at Dunsink.

6. But the photometric curve as a whole is not consistent with the photographic observations. It is to be noticed that this curve is based on a circular orbit and the hypothesis that the stars (ellipsoidal) present uniformly bright discs. Under these circumstances the curve should show a double symmetry. There is in the first place the obvious symmetry about the phase 180°, to which the photographic magnitudes do sensibly conform. In the second place there is a kind of symmetry about the phases 90°, 270°, because the loss of light at corresponding stages of the primary and secondary eclipse should be proportional. The present observations show that this relation is not obeyed. In reality the photometric magnitudes show an analogous discrepancy with the curve, but it occurs on the secondary branch, where it is relatively less conspicuous than on the primary. Something of the difficulty may be explained by an eccentric orbit, with apastron very near the time of primary minimum, as Balanowsky suggests. The determination of the ellipticity of the discs is clearly weak. But it may be surmised that, as in the case of β Lyræ, the hypothesis of discs darkened towards the limb must be introduced if satisfactory light curves are to be obtained. On the analogy of the Sun this effect should be greater in the case of the photographic than for the visual observations. Further examination in detail of this interesting question is withheld for the present, because a more extensive series of photometric observations of TT Aurigæ has been made at Princeton, from which a better determined light curve is expected. The discussion of the present photographic observations may more profitably be undertaken in conjunction with this later and more complete series.

7. The observations are given in Table II. The date with G.M.T. for the middle of each exposure is given in the second column. The third column contains the corresponding Julian day and decimal fraction. The phase in the fourth column is reckoned from the time of preceding primary minimum, calculated with the

epoch and period given in § 3. The fifth column shows the time of exposure, and the sixth the deduced photographic magnitude of TT Aurigæ. The residuals in the seventh column are derived graphically from the photometric curve calculated by J. Balanowsky, as explained in § 4. In the last column are recorded the measurer's estimates of the quality of the images.

TABLE II.

Serial No.	Date G.M.T.	Julian Day.	Phase in days.	Exposure.	Magn.	Obs.–Curve.	Images.
	1914. h m	2420		min			
1	Feb. 19 7 22	183·307	0·569	25	8·28	– ·02	Fair.
2	19 7 48	183·326	0·588	25	8·34	– ·02	Fair.
3	19 8 19	183·347	0·609	25	8·45	+ ·01	Fair.
4	19 8 45	183·365	0·627	25	8·47	– ·03	Fair.
5	19 9 16	183·386	0·648	26	8·60	– ·04	Fair.
6	19 9 42	183·405	0·667	25	8·65	– ·04	Fair.
7	19 10 18	183·430	0·692	30	8·60	·00	Good.
8	19 10 46	183·449	0·711	25	8·52	+ ·02	Fair.
9	19 11 21	183·472	0·734	30	8·42	+ ·04	Fair.
10	23 8 3	187·335	0·599	25	8·41	+ ·02	Fair.
11	23 8 29	187·354	0·618	25	8·48	+ ·01	Fair.
12	23 8 59	187·374	0·638	25	8·59	+ ·01	Fair.
13	23 9 25	187·393	0·657	25	8·67	– ·01	Fair.
14	23 9 58	187·416	0·680	25	8·65	– ·01	Fair.
15	23 10 24	187·434	0·698	25	8·58	+ ·02	Fair.
16	23 11 7	187·462	0·726	30	8·49	+ ·07	Fair.
17	23 11 38	187·485	0·749	30	8·30	– ·03	Fair.
18	23 12 17	187·512	0·776	30	8·25	– ·02	Fair.
19	23 12 51	187·536	0·800	35	8·19	– ·04	Fair.
20	23 13 36	187·567	0·831	35	8·20	·00	Fair.
21	23 14 20	187·597	0·861	35	8·15	– ·02	Fair.
22	23 15 2	187·626	0·890	35	8·14	– ·01	Bad.
23	24 7 20	188·306	0·237	25	8·20	+ ·07	Fair.
24	24 7 46	188·324	0·255	25	8·14	+ ·02	Fair.
25	24 8 18	188·346	0·277	27	8·14	+ ·03	Fair.
26	24 9 3	188·377	0·308	25	8·15	+ ·05	Fair.
27	24 9 35	188·400	0·331	25	8·11	+ ·01	Fair.
28	24 10 0	188·417	0·348	25	8·10	·00	Fair.
29	24 10 33	188·440	0·371	25	8·11	+ ·01	Good.
30	24 10 59	188·458	0·389	25	8·16	+ ·05	Good.
31	24 13 55	188·580	0·511	35	8·17	– ·04	Fair.
32	Mar. 2 7 33	194·315	0·915	20	8·11	– ·02	Fair.

TABLE II.—*continued.*

erial No.	Date G.M.T. 1914.	h	m	Julian Day. 2420	Phase in days.	Exposure. min.	Magn.	Obs. – Curve.	Images.
33	Mar. 2	7	58	194·333	0·933	20	8·13	+ ·01	Fair.
34	2	8	33	194·349	0·949	20	8·08	– ·03	Fair.
35	16	8	4	208·336	0·276	20	8·10	– ·01	Fair.
36	16	8	25	208·351	0·291	20	8·15	+ ·04	Fair.
37	16	8	48	208·367	0·307	17	8·10	·00	Fair.
38	24	9	41	216·404	0·348	30	8·13	+ ·03	Fair.
39	24	10	20	216·413	0·375	30	8·14	+ ·03	Good.
40	Dec. 18	12	52	485·536	0·272	35	8·20	+ ·09	Fair.
	1915.								
41	Jan. 11	10	29	509·438	0·185	40	8·25	+ ·07	Fair.
42	Feb. 15	8	1	544·334	0·430	40	8·20	+ ·07	Fair.
43	15	9	39	544·403	0·499	40	8·20	+ ·01	Fair.
44	15	10	21	544·431	0·527	40	8·17	– ·06	Fair.
45	17	11	22	546·473	1·237	35	8·51	+ ·17	Fair.
46	17	11	58	546·499	1·263	36	8·70	+ ·22	Fair.
47	18	10	18	547·430	0·861	40	8·16	– ·01	Fair.
48	18	11	5	547·462	0·893	40	8·11	– ·03	Fair.
49	Mar. 9	7	53	566·329	1·102	40	8·18	+ ·04	Fair.
50	9	8	35	566·358	1 131	30	8·20	+ ·03	Fair.
51	9	9	7	566·379	1·152	30	8·19	·00	Fair.
52	11	8	33	568·356	0·464	34	8·16	·00	Fair.
53	11	9	3	568·377	0·485	25	8·14	– ·04	Fair.
54	Oct. 15	11	10	786·465	0·008	30	9·15	+ ·01	Good.
55	15	11	41	786·487	0·030	30	9·07	+ ·20	Fair.
56	15	12	19	786·514	0·057	30	8·82	+ ·23	Good.
57	15	12	50	786·535	0·078	30	8·70	+ ·26	Good.
58	15	13	33	786·565	0·108	30	8·40	+ ·09	Good.
59	Nov. 4	10	14	806·427	1·312	30	9·15	+ ·13	Good.
60	4	10	45	806·449	0·001	30	9·12	– ·03	Fair.
61	4	11	22	806·473	0·025	30	9·12	+ ·18	Fair.
62	4	11	54	806·496	0·048	30	8·94	+ ·28	Good.
63	4	12	33	806·523	0·075	30	8·62	+ ·16	Fair.
64	4	13	4	806·545	0·097	30	8·47	+ ·12	Fair.
65	4	13	43	806·573	0·125	30	8·29	+ ·03	Fair.
66	4	14	14	806·593	0·145	30	8·27	+ ·04	Good.
67	29	11	52	831·494	1·057	25	8·10	– ·02	Good.
68	Dec. 7	9	39	839·403	0·970	30	8·10	– ·01	Good.
69	31	9	29	863·396	0·974	30	8·13	+ ·03	Good.

TABLE II.—*continued.*

Serial No.	Date G.M.T. 1916.	h	m	Julian Day. 2420	Phase in days.	Exposure. min.	Magn.	Obs. – Curve.	Images.
70	Jan. 4	8	18	867·346	0·926	30	8·10	– ·02	Fair.
71	4	8	48	867·367	0·947	30	8·15	+ ·04	Good.
72	4	9	22	867·390	0·970	26	8·12	+ ·02	Good.
73	4	10	2	867·418	0·998	30	8·10	·00	Good.
74	4	10	46	867·449	1·029	30	8·10	– ·01	Fair.
75	4	11	19	867·472	1·052	30	8·09	– ·02	Good.
76	4	11	54	867·496	1·076	30	8·14	+ ·02	Good.
77	4	12	25	867·533	1·113	30	8·17	+ ·02	Good.
78	7	6	24	870·267	1·181	17	8·24	+ ·02	Good.
79	7	7	16	870·303	1·217	25	8·36	+ ·08	Good.
80	7	7	56	870·331	1·245	30	8·55	+ ·17	Good.
81	7	8	26	870·351	1·265	30	8·72	+ ·24	Fair.
82	7	9	0	870·375	1·286	30	8·95	+ ·26	Fair.
83	7	9	30	870·396	1·310	30	9·12	+ ·14	Fair.

Dunsink Observatory:
1916 *March* 4.

Mean Areas and Heliographic Latitudes of Sun-spots in the Year 1914.

(*Communicated by the Astronomer Royal.*)

The results here given are in continuation of those printed in the *Monthly Notices*, vol. lxxv. p. 16, and are deduced from the measurements of photographs taken at the Royal Observatories of Greenwich and of the Cape, and in India at the Kodaikánal Observatory and at Dehra Dûn, North-West Provinces.

Table I. gives the mean daily area of umbræ, whole spots, and faculæ for each synodic rotation of the Sun in the year 1914, and Table II. gives the same particulars for each of the entire years 1901 to 1914 inclusive. The areas are given in two forms: first, projected areas, that is to say, as seen and measured on the photographs, these being expressed in millionths of the Sun's apparent disc; and next, areas as corrected for foreshortening, the areas in this case being expressed in millionths of the Sun's visible hemisphere.

Table III. exhibits for each rotation in the year 1914 the mean daily area of the whole spots (corrected for foreshortening), and the mean heliographic latitude of the spotted areas for spots north and for spots south of the equator, together with the mean heliographic latitude of the entire spotted area, and the mean distance

from the equator of all spots; and Table IV. gives the same information for each year as a whole for the years 1901 to 1914 inclusive. Tables II. and IV. are thus in continuation of the similar tables for the years 1874 to 1888 on pp. 381 and 382 of vol. xlix. of the *Monthly Notices*, and for the years 1889 to 1902 on pp. 465 and 466 of vol. lxiii., and for the years 1901 to 1913 on pp. 18 and 19 of vol. lxxv.

The rotations in Table I. and Table III. are numbered in continuation of Carrington's series (*Observations of Solar Spots made at Redhill*, by R. C. Carrington, F.R.S.), No. 1 being the rotation commencing 1853 November 9. The assumed prime meridian is that which passed through the ascending node at mean noon of 1854 January 1, and the assumed period of the Sun's sidereal rotation is 25·38 days. The dates of the commencement of the rotations are given in Greenwich civil time, reckoning from mean midnight.

TABLE I.

No. of Rotation.	Date of Commencement of each Rotation.		No. of Days on which Photographs were taken.	Mean of Daily Areas.					
				Projected.			Corrected for Foreshortening.		
				Umbræ.	Whole Spots.	Faculæ.	Umbræ.	Whole Spots.	Faculæ.
	1913.	d							
806	Dec.	22·53	27	6	27	149	4	20	169
	1914.								
807	Jan.	18·86	27	3	16	137	2	11	151
808	Feb.	15·20	28	0·4	3	91	0·3	2	99
809	Mar.	14·53	27	62	250	255	54	217	295
810	Apr.	10·83	27	60	275	448	44	213	518
811	May	8·07	27	6	18	330	4	12	384
812	June	4·29	28	48	243	529	34	186	580
813	July	1·48	27	12	60	407	10	50	431
814	July	28·69	27	76	423	471	56	301	543
815	Aug.	24·92	27	80	462	617	65	377	728
816	Sept.	21·18	28	11	53	458	10	51	524
817	Oct.	18·47	27	51	202	389	33	137	463
818	Nov.	14·77	27	97	453	684	75	368	788
819	Dec.	12·08	27	76	394	553	51	262	654

[TABLE

TABLE II.

Year.	No. of Days on which Photographs were taken.	Mean of Daily Areas. Projected. Umbræ.	Whole Spots.	Faculæ.	Corrected for Foreshortening. Umbræ.	Whole Spots.	Faculæ.
1901	359	14	41	23	9	29	29
1902	349	14	86	163	10	62	178
1903	350	67	434	875	51	340	970
1904	363	93	653	1639	67	488	1761
1905	364	230	1637	2433	163	1191	2612
1906	364	158	1047	2207	114	778	2320
1907	365	221	1453	1859	160	1082	1999
1908	366	148	952	2000	106	697	2098
1909	364	171	941	1303	124	692	1353
1910	365	67	357	984	49	264	971
1911	364	16	88	458	12	64	459
1912	365	10	50	201	7	37	210
1913	365	2	10	87	1	7	95
1914	365	40	195	396	31	152	454

TABLE III.

No. of Rotation.	Date of Commencement of each Rotation.	No. of Days on which Photographs were taken.	Spots NORTH of the Equator. Mean of Daily Areas.	Mean Heliographic Latitude.	Spots SOUTH of the Equator. Mean of Daily Areas.	Mean Heliographic Latitude.	Mean Heliographic Latitude of entire Spotted Area.	Mean Distance from Equator of all Spots.
	1913. d			°		°	°	°
806	Dec. 22·53	27	12	22·40	7	21·69	+ 5·50	22·13
	1914.							
807	Jan. 18·86	27	0·5	25·00	10	12·49	− 10·79	13·06
808	Feb. 15·20	28	2	14·48	0·5	20·17	+ 5·96	15·88
809	Mar. 14·53	27	217	29·33	0	...	+29·33	29·33
810	Apr. 10·83	27	183	19·70	30	24·27	+13·55	20·34
811	May 8·07	27	11	26·18	0·8	24·06	+22·86	26·04
812	June 4·29	28	19	28·44	168	21·90	− 16·90	22·55
813	July 1·48	27	0·1	26·20	50	32·87	− 32·78	32·86
814	July 28·69	27	290	19·43	11	22·27	+17·84	19·54
815	Aug. 24·92	27	231	18·02	146	21·37	+ 2·74	19·32
816	Sept. 21·18	28	50	19·26	0·9	23·50	+18·53	19·33
817	Oct. 18·47	27	110	22·49	27	20·46	+14·01	22·09
818	Nov. 14·77	27	195	18·17	173	23·93	− 1·60	20·87
819	Dec. 12·08	27	171	16·53	92	22·53	+ 2·87	18·63

TABLE IV.

Year.	No. of Days on which Photographs were taken.	Spots NORTH of the Equator.		Spots SOUTH of the Equator.		Mean Heliographic Latitude of entire Spotted Area.	Mean Distance from Equator of all Spots.
		Mean of Daily Areas.	Mean Heliographic Latitude.	Mean of Daily Areas.	Mean Heliographic Latitude.		
1901	359	22	8°·59	7	16°·27	+2°·82	10°·37
1902	349	42	18·81	21	15·29	+7·48	17·64
1903	350	132	18·11	208	21·15	−5·85	19·94
1904	363	268	16·33	220	16·88	+1·37	16·57
1905	364	750	11·66	440	15·55	+1·60	13·10
1906	364	539	13·98	239	14·01	+5·38	13·99
1907	365	488	10·12	593	13·77	−2·98	12·12
1908	366	316	10·42	381	10·34	−0·92	10·38
1909	364	299	9·45	393	9·92	−1·55	9·71
1910	365	66	8·55	198	11·18	−6·27	10·53
1911	364	17	7·89	47	5·99	−2·30	6·49
1912	365	1	20·53	36	7·78	−7·18	8·06
1913	365	5·0	24·81	2·5	20·08	+9·81	23·23
1914	365	99	21·11	53	23·06	+5·71	21·79
1914	Old Cycle	0·07	3·55	0·05	4·21	+0·37	3·82
1914	New Cycle	99	21·12	53	23·08	+5·72	21·80

The principal features of the record for 1914 are :—

1. The unmistakable and strongly marked revival of solar activity, as regards both spots and faculæ.

2. The results for the separate rotations, and still more the daily results, show that this revival commenced suddenly, its first manifestation in the visible hemisphere of the Sun being the appearance at the east limb of Group 7030 on 1914 March 30. Previous to this date the Sun must be considered as still in the quiescent stage.

3. As usual, the variations from the preceding year have been in proportion much greater for the spots than for the faculæ, the mean daily spotted area for 1914 being more than 20 times as great as that for 1913, while the corresponding value for the faculæ is less than 5 times as great. The spotted area for 1914 equalled that of the year ending 1911 June 30; the faculous area for 1914 that of the year ending 1914 December 31;—six months later. The minimum for spots fell about 1913·5; that for faculæ about 1913·7;—nearly three months later.

4. Comparing the two hemispheres, the predominance in spotted area still remained with the northern hemisphere on the whole, though as the reviving activity had not yet become general and sustained over the whole range in longitude of the spot zones, there

were short periods during the year when the solar activity for a time appeared to be the greater in the southern hemisphere. The great majority of the spot-groups observed were in fairly high latitudes, but six short-lived groups appeared in the equatorial zone, three on each side of the equator. These were so small and short-lived that their inclusion or exclusion makes practically no difference in the mean results for the year, as appears from the figures for the "Old" and "New" Cycles given at the foot of Table IV.

5. The mean distance from the equator of all spots is slightly below that for 1913—21°·79 as compared with 23°·23. This would seem to show that the decline in latitude significant of the first stages of a new cycle has already become apparent.

6. The Sun was free from spots on 143 days in 1914, as against 311 days in 1913, 246 in 1912, and 183 in 1911.

7. The number of separate groups observed in 1914 was 118 as compared with 15 in 1913—an increase of about 8 times. The proportion of small and short-lived spots is therefore decreasing, a further indication of the growing intensity of the solar action.

8. Of the 118 separate groups of the year 57 were in the northern and 61 in the southern hemisphere; in other words, the groups were nearly equally distributed with respect to the equator. This circumstance, so far as it goes, would seem to point to either an early or an ill-defined maximum.

This report has been drawn up for the Observatory by Mr. Maunder.

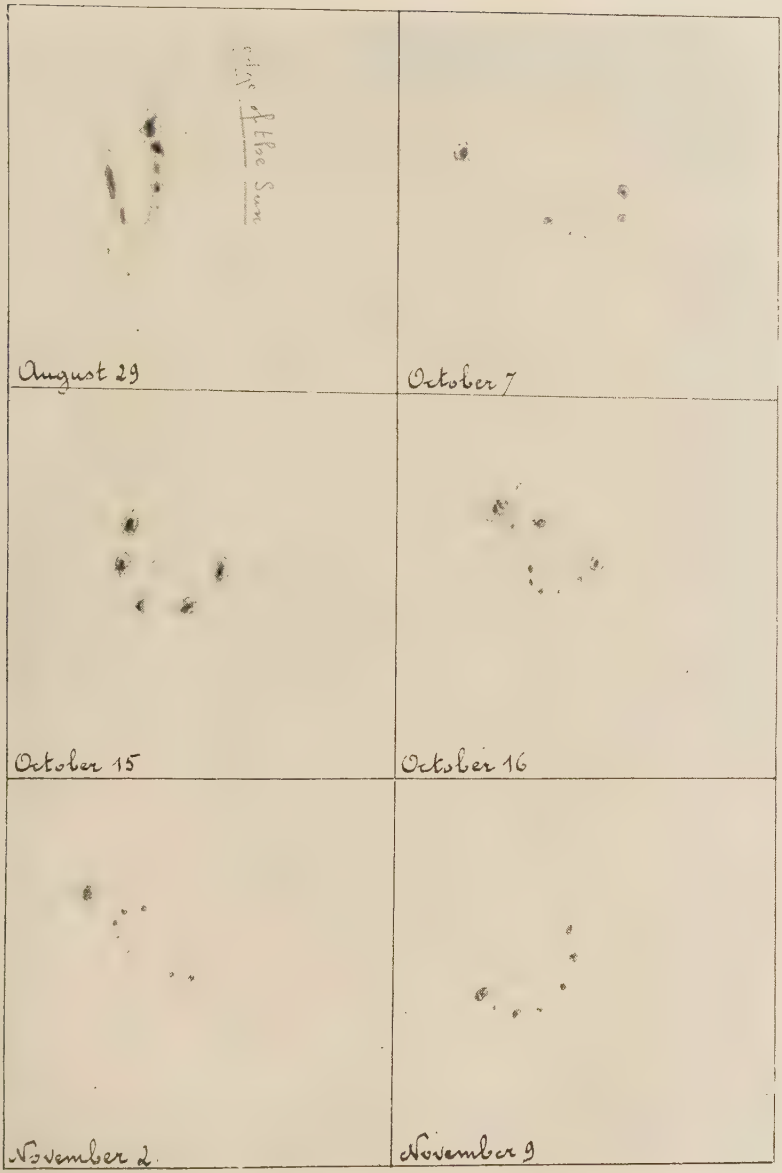
Royal Observatory, Greenwich:
1916 *February* 19.

Some Remarks on the Formation of Sun-spots.

By F. Henroteau, D. ès Sc. (Plate 6.)

From the researches of Professor Hale upon the magnetic fields of Sun-spots it seems highly probable that these latter are due to vortices of matter. This was likewise the hypothesis of Faye, but he supposed, as is held even now by certain astronomers, that the whirling movements were due to the differential velocities in adjoining solar latitudes, according to Carrington's law. But these feeble differences of velocity seem to be insufficient to produce the supposed effect.

In 1912 Professor Hale attributed these vortices either to eruptions or to rapid convection currents. In consequence of differences in rotational velocity, or from irregularities of structure, vortices are formed in these currents and give rise to Sun-spots. There would be a lowering of temperature in the gases constituting these nuclei of disturbance due to their expansion, and hence is



Elliptical Curvature in Groups of Sun-spots. (F. Henroteau.)

explained the formation in the spots of complex chemical compounds (*Pub. Ast. Soc. Proc.*, xxiv. p. 226). My regular observations of the solar surface for more than a year, during a period of greater solar activity, have led me to develop this hypothesis.

According to a valuable research by Father Cortie, based on the study of 3500 drawings of Sun-spots made at the Stonyhurst Observatory, one may conclude that in general a group of spots is composed of two parts, each disposed about a pole. He writes (*Ap. J.*, 1901, xiii. 261): "The chief type, however, of which the above mentioned are in most, possibly in all, cases but phases, is the double-spot formation, with a train of smaller spots between the two principal spots of the group, the whole group generally drifting into more or less parallelism with the solar equator. In this form the principal spot, which eventually becomes a normal spot of regular outline, is generally the leading spot, but in many cases it is the following spot, while sometimes the preponderance in area alternates between the two, as the group traverses the disc. In yet rarer instances both the chief spots develop as regular spots."

This remark concerning the bi-polarity of groups has been confirmed by Professor Hale, who has furthermore discovered that the two parts are of opposite magnetic polarities (*Proc. Nat. Acad. Sc.*, i. p. 382, June 1915). The question naturally arises as to the influence which should be operative in producing such a bi-polar group. In this connection I would wish to draw attention to the frequency of the formation of small spots grouped on an elliptic curve, the nature, and the importance of these formations. I have myself observed a certain number of such formations, and I have found numerous examples among the Stonyhurst drawings of spots during the last thirty-four years. Such formations are also represented in the treatise of physical astronomy of Francis Arago, though he does not call attention to them, in drawings of spots made more than a century ago.

The following are my observations of this phenomenon between the dates 1915 April 1 and December 31.

April 1.—Elliptical chain of spots in the big group.

April 7.—One elliptical group of spots.

April 22.—Some small spots in the southern group are following an elliptical curve.

April 23.—The same phenomenon.

April 29.—Some small spots in the northern group are following an elliptical curve.

May 28.—Some small spots in the north-eastern group are following an elliptical curve.

June 16.—The spots in the equatorial group are following an elliptical curve (see Plate 1 of the *Monthly Notices* of November 1915). The spectroheliograms of Mount Wilson show the curve very beautifully.

June 19.—Some regions on the Sun are very disturbed. The tendency of distribution in circle or ellipse is very evident in two

of the principal regions. The next day the distribution is totally different.

July 26.—The extremity of the big group in the southern hemisphere seems to follow an elliptical curve (see Plate 2, *M.N.*, vol. lxxvi.).

August 18.—Small spots forming elliptical appendices to the northern group (lat. $+15°{\cdot}6$, long. $282°{\cdot}3$).

August 19.—In the same group, spots following more or less an elliptical curve.

August 24.—One of the groups shows the same phenomenon.

August 29.—The whole group of spots (see plate) (lat. $+22°{\cdot}7$, long. $88°{\cdot}6$) is following a distinct elliptical curve. It shows great activity.

September 16.—A kind of current of veiled spots following an elliptical curve is visible in the centre of the disc.

October 7.—One elliptical group of spots (see plate).

October 9.—One elliptical group of spots.

October 13.—Elliptical appendices in certain groups of spots.

October 15.—A fine elliptical group of spots (see plate).

October 16.—A pronounced elliptical current of spots in the same group (still visible on October 17 and 18) (see plate).

November 2.—The disposition in a regular curve is to be noticed in the southern group. Note also that these spots are not to be seen on the drawing of November 1 (see plate).

November 3.—The same group shows the disposition in an elliptical curve, but completely different from that of November 2.

November 9.—A fine group of spots *appears suddenly* following an elliptical curve. The regularity of the curve is striking (see plate). The following days it has become a very important group of spots which shows no further tendency to elliptical distribution.

November 13.—A part of the same group shows elliptical distribution.

December 25.—Some faint dark currents seem to emanate from the spots. They were first visible on the preceding spot, seeming to follow straight lines. Then they appeared near the following spot, seeming to follow a curve (probably elliptical). After a few minutes they disappeared or changed their aspect. Probably they are granulation clusters.

These elliptical currents, which appear the more frequently during periods of greater solar activity, are generally in evidence at the birth of a group of spots, in portions of the surface previously undisturbed. These groups develop in a very short period of time, as, for example, in the case of the group November 9 figured in the plate. I have often seen a group of little spots appear with remarkable rapidity in such an elliptical circuit. Their configuration will then change in a very short time, presenting the appearance of a confused group of small spots. A recrudescence of activity in a group of spots is also marked by these elliptical currents of extreme rapidity of formation. It is not

at all unlikely, were it possible, by a continuous observation of the solar surface, to observe the actual birth of groups of spots, that in numerous cases it would be found that they form in such elliptical currents.

These elliptical currents appear to be the effect of an electrical discharge between two points, or rather two regions, of the Sun's surface charged with electrons of opposite sign. For the rapidity and short duration of these elliptical currents preclude the possibility of their arising from the mechanical or dynamical effect of a material current. It is only after this first discharge that the materials charged with positive electrons at one extremity and with negative electrons at the other are precipitated in the form of a vortex, in the region in which expansion has taken place, as the effect of a kind of explosive action due to the electric discharge. This would explain the bi-polarity observed in the magnetic field of a group of Sun-spots at its two extremities. The expansion of the gas at the centre of these vortices then produces the cooling according to the hypothesis of Professor Hale. The vortices and eddies which follow the discharge, and which constitute the group of spots, could persist for a long time after the discharge had ceased. The electrical discharges which are produced in terrestrial storms have generally a zig-zag form. But in laboratory experiments, with regard to the discharge of statical electricity between two spheres, there are three sorts of discharge in the forms of straight lines, of curved or elliptical lines, and zig-zags. Now, according to the application of the phenomenon of Stark, it is known that there are on the Sun electrical charges of much greater potential than there are upon the earth (*cf.* Salet and Millochau, *C.R.*, clviii. 1914, p. 1000; and Hale and Babcock, *Proc. National Academy of Sciences*, i. 1915, p. 123). Discharges are therefore necessary, and when they take place at a certain distance, they could take the form of elliptical currents, as in laboratory experiments. Such discharges are, of course, only perceptible by means of the groups of spots which are formed by their passage. Moreover, a group of spots would not be formed by a single discharge but by a series of discharges, as in a terrestrial storm. We are thus brought into accord with the theory of Birkeland, who affirms that Sun-spots are due to electrical discharges (*C.R.*, cliii. 1911, pp. 456–459).

One might even dare to go further, and affirm that Sun-spots could not be found in high solar latitudes, because in such latitudes there can only exist electrons of one sign, the electrons of opposite sign being repelled by the solar poles, nor again in general near the solar equator, for there the electrons of contrary sign would neutralise one another. The variation in Sun-spot and solar activity would be nothing else, on this view, but a variation in the total electric potential of the Sun.

Stonyhurst College Observatory:
1916 *January* 7.

The Solar Eclipse of 1917 *December* 13.

By C. T. Whitmell, M.A., B.Sc.

In this paper I consider briefly the central path of the annular solar eclipse of 1917 December 13. For a map of the eclipse the *Nautical Almanac* may be referred to.

Annular solar eclipses occur on the following dates, separated by a Saros. The times refer to centrality, and are in G.M.T. The longitudes are measured from Greenwich.

(1) 1881 November 21. Begins, $3^h\ 42^m{\cdot}2$, long. 176° 35′ W., lat. 51° 34′ S. Central Eclipse at local noon, $4^h\ 42^m{\cdot}3$, long. 74° 2′ W., lat. 84° 24′ S. Ends, $5^h\ 19^m{\cdot}3$, long. 50° 26′ E., lat. 62° 17′ S. Range in longitude, 132° 59′; duration, $1^h\ 37^m{\cdot}1$.

(2) 1899 December 2. Begins, $12^h\ 11^m{\cdot}7$, long. 48° 31′ E., lat. 55° 35′ S. Central Eclipse at local noon, $13^h\ 1^m{\cdot}8$, long. 161° 59′ E., lat. 87° 24′ S. Ends, $13^h\ 43^m{\cdot}2$, long. 75° 48′ W., lat. 59° 14′ S. Range in longitude, 124° 19′; duration, $1^h\ 31^m{\cdot}5$.

(3) 1917 December 13. Begins, $20^h\ 43^m{\cdot}8$, long. 87° 52′·7 W., lat. 59° 1′·9 S. Central Eclipse at local midnight, $21^h\ 23^m{\cdot}4$, long. 142° 12′·8 W., lat. 89° 56′·6 S. Ends, $22^h\ 10^m{\cdot}5$, long. 155° 41′·2 E., lat. 56° 7′·8 S. Range in longitude, 116° 26′; duration $1^h\ 26^m{\cdot}7$.

Considering (1), (2), (3), it will be noticed that, when the Sun and Moon are both on the meridian, the central line of the shadow-track crosses the meridian at points successively nearer the South Pole, and, in fact, for (3) has passed the Pole, for in (3) conjunction in R.A. takes place at local midnight, whereas in (1) and (2) this occurs at local noon.

The foregoing data are from the *N.A.*, those for 1917 being based upon the American Ephemeris. It is rather curious that the *Connaissance des Temps* gives for (3), at conjunction in R.A., the longitude of the shadow as 37° 47′ E., and the time as local noon, thus placing the shadow on the other side of the Pole.

There can, I think, be no doubt that there is an error in the *C. des T.*, and that local noon should be altered into local midnight, with a corresponding change in the longitude. The other French data are in essential agreement with the American.

In connection with polar eclipses, I may point out that in the *N.A.* the annular eclipse of 1891 June 6 is stated to be central at local noon, at $4^h\ 38^m{\cdot}0$, G.M.T., in lat. 71° 11′ S., long. 110° 6′ E. The local time should clearly be midnight. I drew Dr. Downing's attention to this error several years ago, and he expressed surprise that it had escaped the vigilance of Mr. J. R. Hind. I may add that the French Almanac was correct.

We may now consider some details connected with the Central Eclipse of 1917 December 13.

As at conjunction the shadow-axis is stated to meet the Earth's surface (supposed at sea-level) less than four miles from the South Pole (lat. 89° 56′·6 S.), I tested the conditions, using the eccentricity of the terrestrial spheroid employed in the American Ephemeris, viz. 0·0819981.

I find that the axis meets the Earth at a point in lat. 89° 56′·5 S., in practical agreement with the Ephemeris, and that the point falls beyond the Pole in longitude 142° 12′·8 W., so that the time is local midnight.

Denote by C the point where the shadow-axis meets the Earth at local midnight. Let E be the Earth's centre and P the South Pole. Produce the Earth's minor semi-axis, EP = b, to meet the shadow-axis outside the Earth at a point T, and let ET = c; then PT = $c - b$ is only about 9150 feet, or rather less than $1\frac{3}{4}$ miles.

The foregoing calculation assumes, of course, that P is at sea-level, and this raises a very interesting consideration. Recent Antarctic research has revealed to us the fact that around the South Pole is a lofty plateau, some 10,000 feet above sea-level. Mr. Hinks has kindly confirmed this estimate, without going into minute detail.

But, if this is the case, it is obvious that the shadow-axis may be intercepted before it crosses the line of the produced polar axis, and so may actually meet the spheroid at a point in longitude 37° 47′·2 E., on the local noon, instead of on the local midnight, side of the Pole, so that if an observer be situated at this elevated point he may, after all, report that at conjunction central eclipse took place at local noon.

It is, of course, only from the point C, at which the shadow-axis meets the Earth, that the Moon will appear truly concentric with the Sun, either at local noon or midnight, as the case may be.

For the observer at C annularity lasts about $1\frac{1}{4}$ minutes, the annulus, which for this short time remains of constant area, having, when symmetrical, a breadth of only 20″·4, so that the eclipse is a large one, 0·958916 of the Sun's disc being concealed, and only one part out of 24 visible. As the common altitude of the Sun and Moon is about 23°, the eclipse would not be sensibly affected by refraction.

Such an eclipse in such a region would have a grandeur and weirdness quite indescribable, and we can only regret that the chances against its being beheld by human eyes are so enormous.

A total eclipse of the midnight Sun would be a still grander, but a very rare, event. So far as I know, there has not been one since that of 1797 June 24, to which my attention was kindly directed by Dr. Downing. This eclipse was total at local midnight near Cape Chelyuskin, in Siberia, the central track being very similar in position to that of the annular eclipse of 1891 June 6.

Resuming the consideration of the December eclipse, we find that upon a plane through C, parallel to the fundamental plane, the shadow would be a circle about 45 miles in diameter, but on the Earth's surface its shape will be that of a long oval, having a breadth at C of 45 miles, and a length of 115 miles, stretching along the meridian through C and the Pole, and being nearly bisected at C.

From any point within this oval the Sun will appear annularly

eclipsed. Call the ends of the long diameter of the oval M and N, M being in long. 142° 12′·8 W., and N in long. 37° 47′·2 E., then, to an observer between M and P, the eclipse will be reckoned as occurring at local midnight, and, to an observer between P and N, as occurring at local noon. At P itself the time will naturally be indeterminate.

In latitudes so near the Pole, the observer's velocity along his parallel, though opposite in direction to that of the shadow at local midnight, and in the same direction at local noon, will be so small as not appreciably to shorten, or lengthen, the duration of annularity.

Owing to the rapid change of longitude of the shadow in polar regions, amounting to 116° 26′ in about 1½ hours, the G.M.T. and the L.A.T. (local apparent time) of beginning, conjunction, and end phases present interesting contrasts.

	Beginning.			Conjunction.			End.		
		h	m		h	m		h	m
G.M.T.	13th	20	44	13th	21	23	13th	22	11
L.A.T.	13th	14	58	13th	12	0	14th	8	39

Sunrise.	Midnight.	Sunset.
Long. 87° 53′ W.	142° 13′ W.	155° 41′ E.

Let B, C, D be the beginning, conjunction, and end points of the central shadow-track. Then rough measurement on the map makes the arc BCD = 4370 miles, differing but slightly from an arc of a small circle joining B to D, which measures about 4330 miles. The end points, B, D, differ in latitude by only 2° 54′·1. The Sun and Moon are on the horizon for B and D, and at an altitude of 23° for C.

The direction pursued by the track varies from point to point. Briefly, at the start it runs S.E., then S.S.E., then near the time of mid-eclipse and local midnight runs E. to W. It then turns N.N.E. and ends N.E.

The interesting reversal, E. to W., of the usual eclipse W. to E. direction, occurs only over the middle polar part of the track. It seems desirable to point this out, for, if we considered simply the beginning point B, long. 87° 53′ W., and the end point D, long. 155° 41′ E., we might suppose the whole progression to be westward, as it would have been if the shadow had travelled along a great circle from B to D.

A few other details may be briefly indicated. The central shadow-track is a narrow band, 45 miles wide at C, not dark, as is an umbral band, but dimly lit up with an illumination that remains practically uniform at any given locality so long as the annularity is unbroken.

As the South Pole lies within this band, both of its boundaries there are north of the Pole, though one of them corresponds to the north, and the other to the south, limit of the Moon's inner shadow-cone in space.

Had the track fallen in a somewhat lower latitude, so as to lie outside the Pole, then that boundary of the track which was due to the northern limit of the inner shadow-cone, would on the Earth lie nearest to the South Pole, thus illustrating one of the anomalous cases dealt with by Mr. Buchanan in his *Theory of Eclipses.*

Although there is no true umbra in an annular eclipse, the penumbra is divisible into two regions: (1) a circle of small but uniform illumination (at any given time), and (2) a surrounding annulus in which the brightness increases outwards.

There is no generally recognised name for the circular inner area; the term "negative shadow" is not satisfactory. Perhaps it might be called the "semi-umbra."

I have naturally been much interested in Father Rigge's able paper, "The Solar Eclipse of 1916 December 24" (lxxv. p. 687), and should like to say that my notes for this paper were commenced long before his was published, so that any resemblances are undesigned.

Hyde Park, Leeds:
1916 *March* 7.

Recent Observations of Jupiter. By F. Sargent.

During the recent "opposition" Jupiter was observed from Bristol on 90 nights, commencing 1915 May 27, and ending on 1916 February 13.

480 transits were taken, and with a few exceptions these were utilised in deriving the following rotation periods:—

Marking.	Rotation Period. h m s	No. of Spots.	Mean No. of Rotations.
N. Equatorial Belt, N. edge . .	9 55 25·9	19	262
,, ,, ,, S. ,, . .	9 50 17·8	5	99
S. ,, ,, . .	9 55 41·4	1	222
The "Red Spot Hollow" . .	9 55 37·6	1	485
S. Tropical Disturbance, P. end .	9 55 29·1	1	635
,, ,, ,, F. ,, .	9 55 32·5	1	495
,, Temperate Belt	9 55 22·0	2	344

Respecting the observations, I submit the following notes:—

The N.N. Temperate and N. Temperate Belts were frequently seen as faint diffused lines. The former belt on two occasions appeared to terminate, or to present a considerable fall in intensity. The dates were August 4 (in long. 164°, System II.) and September 19 (long. 99°). Any other variation was general, and doubtless due to local conditions.

The N. Equatorial Belt was a most interesting object of study.

In the writer's opinion, it was throughout the darkest belt on the planet, notwithstanding that as the summer progressed the N. edge diminished in intensity, shading off towards the N. Tropical Zone, where the limit of the belt was marked by a series of light and dark spots, frequently linked together by thin dark lines convex towards the south.

Adopting the conventional term "spots" to indicate points capable of repeated identification, 23 spots were observed on the N. edge of the belt, 19 of them sufficiently long to yield satisfactory rotation periods. Of these the average number of observations per spot was 11, and of rotations 262.

On the S. edge of the N. Equatorial Belt there was a revival of activity towards the end of the year 1912. This has now largely spent itself.

During 1915, faint wisp-like markings projecting into the Equatorial Zone were fairly numerous, but generally of a very transient nature, so that identification was difficult. Three dark spots and two white ovals were, however, repeatedly observed, the rotation period derived being as given.

A comparison of the last three years is of interest.

*S. Edge of N. Equatorial Belt.**

Date.	Period. h m s	No. of Spots.	Mean No. of Rotations.
1913	9 50 11·2	11	71
1914	9 50 14·8	11	200
1915	9 50 17·8	5	98

In the S. Equatorial Belt during the early part of October light spots appeared in longitude approximately 135°. One of these persisted, and was still an easy object when observations ceased. During the time it was repeatedly observed, and was judged to have increased in longitude 2° ± according to the System II. rate.

The "Red Spot Hollow" was observed to transit on 27 occasions.

The Red Spot was frequently seen, especially during September. It was not a prominent feature, and uniformly appeared to be situated towards the F. end of the hollow.

During the opposition the region of the R.S.H. was much disturbed, especially at the F. end, where the dark material of the S. Equatorial Belt was frequently crossed by light bands inclined at an angle of approximately 40°, and directed S. following.

The S. Tropical Disturbance has yielded interesting results. In November 1914 the disturbance extended to approximately 112°. When its length was established in July 1915 it was found to be about 83°, being at that time increasing, having passed a

* See *Monthly Notices*, vol. lxxiv. p 116, by Rev. T. E. R. Phillips; *Observatory*, November 1913, by F. Sargent; *ibid.*, January 1915, by F. Sargent.

minimum some time in the interval. By the end of August its length had reached 93°, at which it remained for about two months, when a further distension began, the disturbance being about 105° in length when observations ceased early in February last.

These alterations were almost entirely due to variations at the following end of the disturbance, the average effect at this position being to increase a rotation period already unusually long.

On the S. Temperate Belt two white oval spots were prominent features, and were well observed. One of them was first seen as a notch in the belt on August 13, the other on August 26.

This spot was then in a position due S. of the following end of the Red Spot Hollow, and having the shorter period of rotation, slowly passed along the edge of the hollow. A comparison of its rotation period during this time with that of later date indicates a difference of 1·5 seconds per rotation, the former being the longer period.

The average period for the two spots is strikingly similar, differing by ·7 second only.

The latter spot was still an easy object as recently as February 6.

A few transits of the more prominent features have been taken recently in daylight, but results indicate a tendency to anticipate the true transit by about 3 minutes.

This may be a personal equation, but, being recognised, daylight observations, when conditions permit, are likely to prove useful.

For the observations the instruments used were a 10½-inch reflector, powers 140 and 180, and a 5-inch Cooke refractor, powers 130 and 195.

16 *Cottrell Road, Eastville : Bristol.*

On the Question of Objective Change on the Surface of Mars.

By E. M. Antoniadi.

In a paper sent to the Society in 1908, and published in vol. lxix. pp. 110 to 114 of the *Monthly Notices*, the writer concluded that the surface features of Mars are subject to changes of a remarkable character.

The important part played by cloud in modifying the apparent configuration of the markings on the planet was shown by the observers of the Victorian era. In fact, the white clouds of Mars were discovered by Lockyer in 1862 ;* and in 1879 Burton, with home-made instruments, and extraordinary acuteness, recognised the yellow clouds of the planet.† These results have been con-

* *Mem. R.A.S.*, vol. xxxii., p. 183.
† *Sc. Trans. Royal Dublin Society*, vol. i. (new series), p. 169.

firmed by the writer at Meudon, where the atmospheric veils of Mars, which so frequently deform the dark areas, showed all the scale of tints from brilliant white to deep orange.

Early this year the writer submitted to a careful discussion all the drawings of the planet published since the beginning of the eighteenth century, and concluded that, apart from the modifications due to the interposition of cloud, the surface of Mars undergoes changes of which some are of a secular, others of a recurrent, and others again of a seasonal, character.

1. *Secular Changes.*—Schiaparelli found * that the Hydaspes was very broad and dark to Secchi in 1858, and still easy to Dawes in 1864, whereas this object is usually exceedingly faint or invisible. Further, Van de Sande Bakhuyzen called attention † to the abnormal development of the Cyclops at the close of the eighteenth century, when it appeared most conspicuous and extended to Herschel and to Schroeter. In both these cases we are confronted with the temporary darkening of large ruddy areas.

2. *Recurrent Changes.*—The irregular dusky band Nepenthes–Thoth was very broad and dark in 1888, when Mars was in heliocentric longitude 210° to 256°. It was again most conspicuous in 1911–1912, between heliocentric longitudes 35° to 95°. It was further a strong marking in 1913–14, when Mars passed from heliocentric longitude 93° to 114°; and also on 1916 March 1, in heliocentric longitude 149°. Hence the recurrent, but not seasonal, modifications of this band.

3. *Seasonal Changes.*—The variations in breadth of Syrtis Major seem almost subordinated to the distance of Mars from the Sun. Thus that dusky area was broad after aphelion in 1858, 1873, 1890, and 1905, but narrow after perihelion in 1862–1864, 1877, 1892, and 1909. The maximum narrowness appears to occur at about 15° of heliocentric longitude; after 70° ± of longitude the Syrtis expands to the left, attaining its maximum breadth in longitude 230° ± ; and there is a somewhat rapid shrinkage towards heliocentric longitude 290°.

The encroachment of Syrtis Major on Libya and Isidis Regio between 1877 and 1890 was detected by Schiaparelli. ‡

On the other hand, change in the appearance of the dark spots on Mars was first recognised by Secchi in 1858. §

* *L'Astronomie*, 1889.
† *Memoir on the Rotation Period of Mars*, 1885.
‡ *L'Astronomie*, 1889.
§ *Osservazioni di Marte*, 1858.

74 *Rue Jouffroy, Paris:*
1916 *March* 4.

The Nature of the Coronium Atom. By Professor J. W. Nicholson, M.A., D.Sc.

In earlier papers * on the constitution of the solar corona it was shown that the greater part of its spectrum can be interpreted in terms of the vibrations of the atom of a hypothetical substance to which the name protofluorine was assigned. If e is the charge on an electron, the positive core of the atom, or the so-called "nucleus," has a charge $5e$, and in the normal steady state, when uncharged as a whole, the atom contains five electrons circulating round this nucleus. The angular momenta of electrons follow definite laws within the atom, which are now known so far as to admit the calculation of the spectra of certain charged atoms from those of others.† The masses of nuclei, or atomic weights, in these simple-ring systems are proportional to the squares of the nuclear charges, and any single atom can emit a spectrum which consists of several series of lines. The cube roots of the wave-lengths in any one series form an arithmetic progression, and the members of one series are in a definite constant ratio to the corresponding members of the others. Expressed by a formula, the wave-lengths are of the types

$$\lambda = (a_1, a_2, \ldots)(n+\mu)^3$$

where a_1, a_2, . . ., and μ are constants whose theoretical values are known. The investigation of the spectrum of protofluorine failed to take account only of five lines whose existence in the coronal spectrum is certainly known. One of these is the green line λ 5303·3, whose unknown origin has always been designated as *coronium* by astrophysicists. The others are $\lambda\lambda$ 4566·0, 4359·0, 3642·5, 3534. Although these lines cannot be ascribed to protofluorine, it was pointed out that they appeared to follow similar series laws, for $\lambda\lambda$ 5303·3, 4359, 3534 have constant differences between their cube roots.‡ Thus

$$(5303{\cdot}3)^{\frac{1}{3}} - (4359{\cdot}0)^{\frac{1}{3}} = 1{\cdot}103,$$
$$(4359{\cdot}0)^{\frac{1}{3}} - (3534{\cdot}0)^{\frac{1}{3}} = 1{\cdot}103.$$

But as only three lines were concerned in this relation, which might therefore be accidental, further information was required before the relation of coronium to protofluorine and the constituents of nebulæ could be discussed.

At the recent eclipse visible in Russia, MM. Deslandres and Carrasco independently obtained good photographs of a new and strong line in the red, and their measurements agree in its wave-length, which may be taken as $\lambda = 6374{\cdot}5$.§ This line, like the

* *Monthly Notices*, lxxii., 1911–12, pp. 139, 677, 729.
† *L.c.*, and *Monthly Notices*, lxxiv., 1913–14, p. 486.
‡ *Monthly Notices*, lxxii., 1911–12, p. 739.
§ *Comptes Rendus*, 1914 Nov. 16, p. 669 (Deslandres); *ibid.*, 1914 Nov. 30, p. 740 (Carrasco).

others, cannot be obtained from the formulæ which give the protofluorine spectrum, but M. Carrasco, in a recent note,* has shown that it is the next member of the cube-root series already suggested. Thus

$$(6374{\cdot}5)^{\frac{1}{3}} - (5303{\cdot}3)^{\frac{1}{3}} = 1{\cdot}103,$$

the agreement being very exact. This confirmation is striking, and there can be little doubt that the cube-root series put forward by the writer two years earlier is a reality. It is now possible to proceed with some confidence to the discussion of the nature of the coronium atom.

The wave-length of the next member of the series is given by

$$\lambda^{\frac{1}{3}} = (3534{\cdot}0)^{\frac{1}{3}} + 1{\cdot}103,$$

whence

$$\lambda = 2820;$$

and if observations in this region were possible, this line would probably be found. The next member of the series on the red side is found to be $\lambda = 7581{\cdot}5$.

Of the six coronal lines not yet interpreted, including the new red line, four are therefore the visible members of the series

$$7581{\cdot}5,\quad 6374{\cdot}5,\quad 5303{\cdot}3,\quad 4359{\cdot}0,\quad 3534,\quad 2820,\ \ldots$$

and the other two are $\lambda\lambda$ 4566·0 and 3642·5, which must be brought into relation with this series.

The relation can be found at once, for

$$\frac{4566{\cdot}0}{3534{\cdot}0} = 1{\cdot}292, \qquad \frac{3642{\cdot}5}{2820} = 1{\cdot}292.$$

This coincidence is remarkable, especially in view of the fact that we are dealing with the last remaining lines in the spectrum. The form of the relation indicates, moreover, that $\lambda\lambda$ 4566·0 and 3642·5 are successive members of another cube-root series in a constant ratio to the first—the type of relation required by analogy with the protofluorine spectrum.

It has now become evident that all the remaining lines not due to protofluorine can arise from a very similar simple-ring element, and it only remains to determine the strength of its nucleus and the number of electrons in its ring when it is emitting this spectrum.

Now the two main vibrations about any steady state of a simple-ring atom with nucleus ve of mass M, and n electrons each of mass m, have wave-lengths in the ratio †

$$\left(\frac{\nu}{\nu - \frac{1}{4}S_n'}\right)^{\frac{1}{3}}\left(1 + \frac{n}{2}\cdot\frac{m}{M}\right)$$

* *Comptes Rendus*, 1915 Nov. 22, p. 631.
† *Monthly Notices*, lxxii., 1911–12, p. 683.

where m/M is small. We write therefore

$$\left(\frac{\nu}{\nu - \frac{1}{4}S_n}\right)^{\frac{1}{2}} = \frac{4566{\cdot}0}{3534{\cdot}0} = 1{\cdot}29202$$

where

$$S_n = \sum_{r=1}^{n-1} \operatorname{cosec} \frac{r\pi}{n},$$

with values

$$S_3 = 2{\cdot}30940, \quad S_4 = 3{\cdot}82843, \quad S_5 = 5{\cdot}50552,$$
$$S_6 = 7{\cdot}30938, \quad S_7 = 9{\cdot}21906, \quad S_8 = 11{\cdot}21946,$$
$$S_9 = 13{\cdot}29332.$$

The angular momentum law, which limits the number of possible simple-ring systems,* indicates that the higher values of S_n cannot be used. Now, neglecting m/M,

$$\frac{\nu}{\nu - \frac{1}{4}S_n} = (1{\cdot}29202)^2$$

and

$$\frac{S_n}{\nu} = 1{\cdot}6028.$$

Comparing with $\frac{S_8}{7} = 1{\cdot}60278$, it is evident that $\nu = 7$, $n = 8$, for no other values are even approximately possible, and this value is remarkably accurate.

Now the mass M is to that in a nebulium atom in the ratio $7^2/4^2 = 49/16$. Quoting the mass of the nebulium atom,† m/M is in the present case $0{\cdot}000065$. The ratio of the wave-lengths when corrected should therefore be

$$\left(\frac{7}{4{\cdot}195135}\right)^{\frac{1}{2}}(1{\cdot}00026) = 1{\cdot}29207$$

against the actual value

$$\frac{4566{\cdot}0}{3534{\cdot}0} = 1{\cdot}29202.$$

The agreement is almost absolute, and well within the limit of experimental error. Such an agreement in the case of the final lines in a fairly long spectrum is very convincing, and we deduce that coronium is the simple-ring system with nucleus $7e$. When it has eight electrons, or a single negative charge, it emits the lines

$$\lambda\lambda\, 6374{\cdot}5, \quad 5303{\cdot}3, \quad 4566{\cdot}0, \quad 4359{\cdot}0, \quad 3642{\cdot}5, \quad 3534{\cdot}0.$$

The fact that it is capable of taking negative charges, and that all its lines which are visible are due to negatively charged atoms,

* *Monthly Notices*, lxxiv., 1913–14, p. 488.
† *L.c., ante.*

is in accord with a somewhat similar result for protofluorine, and gives further evidence of a strong negative charge in the corona.

Further examination by the angular momentum principle, by means of which we can deduce the spectra of neutral and of positively charged atoms, shows that all their lines are in these cases too far in the ultra-violet for observation. The visible spectrum is therefore limited to the lines already dealt with.

King's College, London:
1916 *March* 7.

The Spectrum of γ Argûs. By W. M. Worssell.

A prismatic camera has recently been erected at the Union Observatory, Johannesburg. The object glass is a Cooke triplet of 6 inches aperture and about 108 inches focal length, and the prism has a refracting angle of about 16°. The instrument was the gift of the late Mr. J. Franklin-Adams.

As so much attention has already been directed to the blue part of the spectrum, Mr. Innes considered that the most useful programme of work for the instrument would be to attempt photographs in the green, yellow, and red regions. Wratten & Wainwright's panchromatic plates were accordingly employed, and a number of stellar spectra of different types extending to H_α, or beyond, have been photographed.

On my recent arrival in London, some of the negatives were submitted to Professor Fowler, who considers the results very encouraging, and thinks that an account of the spectrum of γ Argûs may be of interest.

γ Argûs (mag. 3) is well known as being by far the brightest member of the class of Wolf-Rayet stars. The visual part of the spectrum has previously been described by Campbell,* and photographs extending as far as the yellow were also obtained by McClean,† and by the Harvard‡ observers at Arequipa. Two photographs were taken at the Union Observatory, one on 1915 April 8 and the other on 1915 April 29; the details are evidently identical on the two plates, but that of April 8 is not so sharply focussed as the other. Because of the long exposure required for the less refrangible parts of the spectrum the blue is much over-exposed, and the present note necessarily refers almost entirely to the less refrangible side of H_β. Professor Fowler has very kindly measured the plate of April 29, and has prepared the account which follows.

The bright lines are mostly very broad, and if it were not for the fact that some of them show very distinct reversals, reasonably approximate measures would scarcely have been possible. In accordance with the observation of Campbell, H_α is very bright,

* *Astron. and Astrophysics*, vol. xiii. p. 468 (1894).
† *Spectra of Southern Stars* (London: 1898).
‡ *Harvard Annals*, vol. xxviii. p. 243.

and is not certainly reversed ; while H_β, though not much brighter than the background of continuous spectrum, is clearly reversed, and succeeding hydrogen lines appear to be wholly dark. The most striking case of reversal, though it has not been noted before, is that of the D_3 line of helium ; at first sight this presents the appearance of a fairly well-separated pair of bright lines, but comparison with other photographs of helium stars leaves no doubt that the dark space between the two components is really coincident with D_3. Some of the other bright lines certainly also show reversals, but in other cases reversal is doubtful.

For the determination of wave-lengths the reference lines were H_α, D_3, and H_β, these being identified by comparison with other spectra. A check on the interpolation formula was afforded by the line at 5412, which agrees in position with the corresponding member of the "Pickering" series shown in a photograph of ζ Puppis. The general accuracy of the reduction is also indicated by the satisfactory agreement of other lines with the determinations of Campbell. No great accuracy can be claimed for the wave-lengths, however, as the average linear dispersion between H_α and H_β is only 1 mm. = 130 A.U. The accuracy is, of course, likely to be greatest in the case of the lines which are clearly reversed, and in these cases the wave-lengths should not be more than 1 or 2 A.U. in error. For the weak lines, in which reversal is not clear, the error may amount to several A.U. An attempt has been made to measure the widths of the brighter lines, but these are only to be regarded as roughly approximate.

The results for the bright bands are given in the appended table. The wave-lengths in the first column are those of the absorption lines in the case of reversed lines, and of the estimated middle points in the case of lines not clearly reversed. The intensities and widths are shown in the second and third columns. Campbell's results are given in the fourth and fifth columns, the scale of intensity in this case being + + = very bright ; + = bright ; unmarked = easily measurable ; − = difficult ; three-figure wave-lengths refer to lines which were too faint to justify micrometer measures.

Bright Lines in γ Argûs.

Wave-length.	Int.	Width in A.U.	Campbell. λ.	Campbell. Int.	Remarks.
6729	1	25			
6676	1	25	...	...	Probably He 6678.
6585	4	16			
6563	7	20	6565	+	H_α.
6460	< 1	...	...	...	Vague broad band.
6407	< 1	...	...	...	Vague broad band.
5876	4	24	5874	+	Helium D_3, strongly reversed.
5811	10	20	5813	+ +	Reversed. Assigned to carbon by Merton.

Bright Lines in γ Argûs.—continued.

Wave-length.	Int.	Width in A.U.	Campbell. λ.	Campbell. Int.	Remarks.
5750	<1	30	...	...	Rather vague, possibly reversed.
5694	10	30	5694	++	Reversed. Assigned to carbon by Merton.
5622	<1	...	...	...	A vague brightening about here.
5598	<1	...	5596	–	Rather vague. Assigned to carbon by Merton.
5522	<1	...	...	...	A vague brightening about here.
5467	2	16	5470	..	Feebly reversed.
5412	2	16	5411	–	Feebly reversed. "Pickering" (ζ Puppis) line.
5360	<1	...	...	...	Vague broad band.
5268	<1	...	...	....	Vague broad band.
5128	1	18	5135	–	Ill-defined, probably reversed.
5017	2	16	5020	–	Helium 5017·86; feebly reversed.
4922	2	16	...	...	Helium 4922·11; feebly reversed.
4862	3	20	...	...	H_β, reversed.

In addition to the bright lines shown in the table, there are indications of dark lines, unaccompanied by bright fringes, at wave-lengths 5851 and 5780.

Of the tabulated bright lines not recorded by Campbell, that at 6583 is probably identical with the line 6583·4 which has been observed by Merrill * in "the hydrogen envelope" of Wolf-Rayet star B.D. + 30° 3639 and by Wright † in some of the nebulæ. The associated line 6548·5 observed by Merrill and Wright, however, does not appear, but it may possibly be merged into H_α in the photograph.

The line at 6729 is probably identical with a line at 6730 observed by Wright in B.D. + 30° 3639 and in some of the nebulæ. This and the line following at 6676 give the appearance of a very broad reversed line, with reversal at 6705, but in view of the presence of so many lines of helium in γ Argûs it is probable that they should be regarded as two separate bright lines, as in the table, one being helium 6678. A similar remark applies to the two bands 6460 and 6407, which look like one very broad bright band with a reversal about 6438.

The helium line 4922 not given by Campbell in any Wolf-Rayet star was confirmed by direct comparisons with the photographs of helium stars. It also appears in the list of lines given by the Harvard observers. Campbell gives 4940 as the mean wave-length of a line in 15 stars.

The broad band at 5268 may possibly correspond to a blending

* *Lick Obs. Bull.*, No. 230, vol. vii. p. 129.
† *Astrophysical Journal*, vol. xl. p. 470.

together of two faint bands recorded by Campbell in some of the Wolf-Rayet stars at 5284 and 5250.

On the more refrangible side of H_β the brightest lines are those at 4686 and 4652, observed by Campbell and others, the latter being the brighter. Apart from the lines of hydrogen and helium, and the lines attributed to enhanced carbon by Merton,* no satisfactory origins can at present be assigned.

Although the Johannesburg plates show a greater number of bright lines than were tabulated by Campbell, the brighter bands are evidently identical in the two cases, and there is no sufficient reason to suppose that the spectrum has changed in the twenty-one years separating the observations.

The Distribution in Space of the Stars in Zone +25° *of the Oxford Astrographic Catalogue.* By R. J. Pocock, B.A., B.Sc.

I. *Introductory.*

Much work has been done recently on the distribution in space of groups of stars as determined from their proper motions. The work has been for the most part confined to bright stars, there being little data for the faint stars. One of the most complete investigations of proper motions among faint stars has been carried out at Oxford and published in a number of papers; in the present investigation the results contained in the "Fifth Note on the Number of Faint Stars with Large Proper Motions"† have been used. This paper contains over 400 proper motions not less than 8″·8 per century.

It is not possible to discuss these motions in relation to spectral type, but this will not affect the comparison with the Astronomer Royal's first paper on the stars in Carrington's Circumpolar Catalogue.‡ In particular it is desirable to test the agreement of theory and observation for the large proper motions among faint stars, since the number of such proper motions used in the investigation of the Carrington stars was small.

II. *Theory.*

According to the general theory as to the distribution of velocities, the number of proper motions in a direction in which there is no stream velocity nor component of the solar motion, which lie between τ and $\tau + d\tau$, is given by the expression

$$N \cdot \frac{d\tau}{a} \cdot \left\{ 1 + \frac{\tau^2}{a^2} \cdot \right\}^{-\frac{\lambda}{2}-1} \quad . \quad . \quad . \quad (1)$$

* *Proc. Roy. Soc.*, A, vol. xci. p. 498.
† *M.N., R.A.S.*, 1911 Nov., p. 65.
‡ *M.N., R.A.S.*, 1913 March, p. 334.

where N is the total number of stars, and λ, a are constants of which λ is known to be near unity.

If τ is large compared with a, we may write this expression as

$$N \cdot a^2 \cdot \frac{d\tau}{\tau^3} \quad . \quad . \quad . \quad . \quad (2)$$

If, however, there is a stream velocity V, the expression for large values of τ is

$$N \cdot \frac{a^2 \cdot d\tau}{\tau^3} \cdot (1 + 2h^2 \cdot V^2 \cdot) \quad . \quad . \quad . \quad (3)$$

which reduces to (2) when $V = 0$.

Also in the two cases the numbers numerically greater than τ are given by

$$N \cdot \tfrac{1}{2} \cdot \left(\frac{a}{\tau}\right)^2 \quad . \quad . \quad . \quad . \quad (4)$$

and by

$$N \cdot \frac{a^2}{\tau^2} \cdot (\tfrac{1}{2} + h^2 \cdot V^2 \cdot) \quad . \quad . \quad . \quad (5)$$

which reduces to (4) when $V = 0$. Also $a = \frac{k}{h}$ where h, k are two constants. Hence, whether there is a stream velocity or not, the number of proper motions in the range τ to $\tau + d\tau$ varies as $\frac{d\tau}{\tau^3}$, and the number numerically greater than τ varies as τ^{-2}.

Assuming N and hV to be known, and adopting $a = 1{\cdot}4$ and $\lambda = 1$ as a first approximation, we can compute the values of $N \cdot \frac{d\tau}{1{\cdot}4} \left\{ 1 + \frac{\tau^2}{(1{\cdot}4)^2} \right\}^{-\frac{3}{2}}$, and the result should agree with the observations save for a constant factor. If we divide this factor by the value of $(1 + 2h^2 \cdot V^2)$, the result will give us the square of the ratio of the true and adopted values of a, and if our approximation is good this ratio should be nearly unity.

To find the factor we must make the total number of proper motions within certain limits agree in theory and observation. The results are still true even if we sum several regions for which the values of a and hV are different.

If T is the algebraic mean of the values of τ, we have

$$\mu \cdot T = k\sqrt{\pi} \cdot V \cdot \quad . \quad . \quad . \quad . \quad (6)$$

where μ is a constant introduced to reduce T and V to the same units. Also $k\sqrt{\pi}$ is the mean parallax of these stars. Also if n_- and n_+ are the total numbers of negative and positive proper motions, we have *

$$\frac{n_-}{n_- + n_+} = \frac{1}{\sqrt{\pi}} \cdot \int_{hV}^{\infty} e^{-x^2} dx \quad . \quad . \quad (7)$$

* See *M.N.*, *R.A.S.*, 1915 March, p. 351, where this formula is given by Professor Eddington, but there appears to be an error in the computation of hV for his region LK. The value should be $-{\cdot}05$, not $-{\cdot}12$; the error is of no consequence, since this value has not been used in determining h.

from which hV can be determined. And

$$\frac{\mu \,.\, \mathrm{T}}{h\mathrm{V}} = \frac{k\sqrt{\pi}\,.\,\mathrm{V}}{h\mathrm{V}} = a\sqrt{\pi} \quad . \quad . \quad . \quad (8)$$

Hence a being known, we can determine μ. If we use several groups of stars with different values of a, T, hV, the agreement of the values of μ for different groups forms a good criterion of the accuracy of the results.

For large values of τ the correction for the probable error of the observations is negligible.*

The constant a can also be determined from (5) by using a particular value of τ and comparing with the observations.

III. *The Values of N, λ, a.*

N denotes the total number of stars used, and from the paper in question we find that the total number of stars compared was 16,112; but since there were 51 overlapping plates the actual number of stars was really smaller, and a correction to this number has been calculated and the resulting number N = 14,941 has been used.

As regards the value of λ, Professor Eddington found λ = 1, and the Astronomer Royal found $1{\cdot}0 > \lambda > 0{\cdot}8$, so that the value λ = 1 has been adopted in the investigation for simplicity; but the observations themselves lead to this value. Thus if τ_1, τ_2 are large, the ratio of the number of proper motions greater than τ_1 to the number greater than τ_2 is $\left(\frac{\tau_2}{\tau_1}\right)^{\lambda+1}$. From the last two figures in column 5, Table I., we find $\frac{10{\cdot}6}{13{\cdot}4} = \left(\frac{9}{10}\right)^{\lambda+1}$, from which $\log(\lambda+1) = 0{\cdot}3009$, and therefore λ = 1·0.

The value of a adopted in calculating the value of the expression (1) is 1·4. It will be seen later that the actual value lies between 1·5 and 1·7, but it did not seem worth while making a fresh calculation.

IV. *The Determination of hV.*

In determining hV the proportion of negative proper motions determined from those stars only which have a large resultant proper motion has been used. The component proper motions, however, are in many cases quite small. If all the stars in the group had been used, the proportion would have been correctly given by the formula (7).

If, however, we only use large components, the proportion is found from the equation

$$n_-' \backsim n_+' = \mathrm{N} \cdot \frac{k^2}{h^2\,.\,\tau^2} \cdot \left\{ \left(\tfrac{1}{2} + h^2\,.\,\mathrm{V}^2\right) \frac{2}{\sqrt{\pi}} \cdot \int_0^{h\mathrm{V}} e^{-x^2} dx + \frac{h\mathrm{V}}{\sqrt{\pi}} \cdot e^{-h^2\mathrm{V}^2} \cdot \right\}$$

* See *M.N., R.A.S.*, 1913 March, p. 359.

combined with equation (7) above, where n_- and n_+ are replaced by n_-' and n_+', n_-' denoting the number of negative proper motions numerically greater than τ. Hence $\frac{n_-'}{n_-'+n_+'}$ is independent of τ as long as τ is large. Also

$$\frac{n_-'}{n_-'+n_+'}=\frac{n_-}{n_-+n_+}-\frac{hV\,.\,e^{-h^2V^2}}{\sqrt{\pi}\,.\,(1+2h^2V^2.)}\,.$$

This last term increases as hV decreases; but when hV is small it is of little use for determining h. If we put $hV=-0\cdot33$, which is approximately its mean value, we find

$$\frac{hV\,.\,e^{-h^2V^2}}{\sqrt{\pi}\,.\,(1+2h^2V^2)}=\frac{0\cdot33\times0\cdot90}{1\cdot77\times1\cdot2}=0\cdot14.$$

This is a considerable quantity compared with the value of $\frac{n_-}{n_-+n_+}$; but, as already pointed out, we have not used exclusively large components; of the components used only about half are greater than $9''\cdot0$ per century, the remainder ranging from zero to this value.

When (as usually in this paper) hV is negative the ratio used is too large, and hV is numerically too large by about the same amount. In the case of the η components, if we used the value $hV=-0\cdot18$ instead of $hV=-0\cdot23$, we should have $a'=1\cdot55$ instead of $a'=1\cdot52$. For large values of hV the correction to a' will be smaller: smaller values of hV are in any case of little value.

V. *Comparison with the Observations.*

From the methods adopted at Oxford the number of small proper motions will be in defect of the true number, especially where the interval is short. For the larger proper motions, however, we may expect the results to be reasonably complete; and although some of the individual proper motions may not be well determined, the numbers within given limits are probably highly accurate.

In the discussion the material is divided into eight three-hour groups, with centres at $+25^\circ$, $1^h\ 30^m$, $4^h\ 30^m$, etc.

The numbers of components between certain limits are given in Table I. The first column gives the limits of proper motion, the second, third, and fourth columns give the numbers of components along the axes of η, x, y; x and y referring to R.A. and Declination and the η axis (which does not differ much from the y axis) being perpendicular to the direction of streaming. The actual numbers have, however, in each case been multiplied by a factor to make the total numbers between 40 and 100 the same in

columns 2, 3, 4, and 6. Column 5 is the mean of 3 and 4, and column 6 gives the values of the expression

$$14{,}941 \cdot \frac{d\tau}{1{\cdot}4} \cdot \left\{ 1 + \frac{\tau^2}{(1{\cdot}4)^2} \cdot \right\}^{-\frac{3}{2}}.$$

For the proper motions between 30 and 40 we need not expect a good agreement; the differences between columns 2, 3, 4 give some idea of the accidental differences to be expected.

TABLE I.

Numbers of Components of Proper Motion between Certain Limits.

Limits of P.M. (Unit 0″·3 per Century.)	Reduced Nos.			Mean of x and y.	Calculated.
	η Comps.	x Comps.	y Comps.		
30					
	56·5	65·8	47·7	56·8	88·9
40					
	37·5	37·4	35·9	36·6	40·6
50					
	29·5	28·4	29·7	29·0	21·4
60					
	8·6	14·2	8·7	11·4	12·9
70					
	10·4	6·7	9·9	8·3	8·3
80					
	4·3	3·0	6·2	4·6	5·6
90					
	2·5	3·0	2·5	2·8	4·0
100					
	8·0	12·0	9·3	10·6	17·1

Allowing for the accidental differences, the agreement between theory and observation is good, except that the numbers of very large proper motions (over 80 or 90) are uniformly in defect of the theory. This fact may be exhibited in another way; from (4) and (5) we see that the numbers numerically greater than τ when τ is large vary as τ^{-2} for any given group. Summing for all the groups this is still true.

In Table II. the first line gives the proper motion, the second the sum of the numbers of x and y components greater than the corresponding number in the row above, and the third row is proportional to the product of the second row and the square of the first row.

TABLE II.

x *and* y *Components compared with Theory.*

Centennial P.M. (Unit 0″·3) .	30	40	50	60	70	80	90	100
Sum of x and y components .	470	305	197	111	78	53	39	31
Product	423	488	492	400	382	339	316	310

It is clear that the proportion of large proper motions is too small.

In Table III. are given particulars of the η components for each group.

TABLE III.

η Components. Data for each Group.

Group	1	2	3	4	5	6	7	8	Mean
Galactic latitude	−36°	−15°	+18°	+58°	+78°	+40°	+20°	−27°	
No. of stars	867	3013	2419	1120	1177	1498	3400	1532	
Ratio $\frac{n_-}{n_- + n_+}$	0·66	0·68	0·82	0·58	0·51	0·57	0·72	0·69	
hV	−0·30	−0·33	−0·65	−0·14	−0·02	−0·13	−0·41	−0·35	−0·29
a	1·06	1·51	0·84	1·93	2·47	1·78	1·06	1·35	1·50
T	−14	−16	−39	−6	−9	−2	−21	−20	
μ	0·057	0·055	0·045	0·062	(0·007)	(0·180)	0·051	0·046	0·052

The values of a agree fairly well, and the mean $a = 1{\cdot}50$ justifies the use of the value $a = 1{\cdot}4$ in the denominator of (1). On the whole they vary with galactic latitude. The values of μ also agree well: in determining T all proper motions were used, and some of the values of T are probably affected by the presence in a group of one or two very large proper motions. This cause is probably sufficient to explain the slight variations in μ; the two cases in brackets are clearly not well determined (in one case hV and in the other T is small).

The factor used to obtain column 2 of Table I. from the actual numbers was 0·615. Hence if a' is the true value of a and $(1 + 2h^2 . V^2)$ has its mean value

$$0{\cdot}615 = \left\{ \left(\frac{a'}{a}\right)^2 . (1 + 2h^2 V^2 .) \right\}^{-1}$$

from which $a' = 1{\cdot}61$.

The total number of negative proper motions is 270, and the total number both positive and negative is 432. Hence $\frac{n_-}{n_- + n_+} = \frac{270}{432} = 0{\cdot}625$ and $hV = -0{\cdot}23$, and the two methods mentioned in II. of determining a give $a' = 1{\cdot}52$ and $a' = 1{\cdot}70$. The former method of the two is probably the more reliable.

In Table IV. are given the same data for the x, y components, but the galactic latitude and total number of stars in a group are not repeated from Table III.

TABLE IV.

x and y Components. Data for each Group.

Group *x* Comps.	1	2	3	4	5	6	7	8	Mean
Ratio $\frac{n_-}{n_- + n_+}$	0·33	0·57	0·51	0·64	0·58	0·51	0·59	0·37	
hV	+0·31	−0·12	−0·02	−0·25	−0·14	−0·02	−0·16	+0·24	
a	1·98	1·26	1·14	2·09	2·86	1·86	1·02	1·49	1·71
T	+21	−5	−4	−18	−8	−12	−4	+12	
μ	0·052	0·054	(0·010)	0·052	0·089	(0·005)	0·072	0·052	0·062

TABLE IV.—x *and* y *Components. Data for each Group*—continued.

Group . . y Comps.	1	2	3	4	5	6	7	8	Mean
Ratio $\frac{n_-}{n_- + n_+}$.	0·69	0·68	0·91	0·62	0·55	0·56	0·69	0·71	
hV . . .	−0·35	−0·33	−0·95	−0·22	−0·09	−0·11	−0·35	−0·39	
a . . .	1·71	1·58	0·79	1·75	2·54	1·68	1·31	1·98	1·67
T . . .	−17	−15	−41	−9	−12	−1	−27	−22	
μ . . .	0·062	0·061	0·033	0·075	0·033	(0·330)	0·030	0·059	0·050

The variation of a with galactic latitude is again in evidence. The values of μ are not in quite such good agreement as in Table III., but the errors appear to be accidental, and the mean is about right.

The factors used in Table I. for the x, y components are 0·748 and 0·619, from which $a' = 1·56$ and 1·52, which we may compare with the means 1·71, 1·67 above.

Using, however, all the stars as before, we have $\frac{n_-}{n_- + n_+} = \frac{230}{432} = 0·532$ for the x components and $= \frac{279}{432} = 0·646$ for the y components. $\therefore$ hV $= -0·06$ and $-0·27$, from which the two methods give $a' = 1·64$ and 1·61 for the x components and $a' = 1·67$ and 1·67 for the y components. The values of T have been compared with those obtained from zone + 29°, though the latter depend on fewer stars.*

The only value of T which appears to be seriously in error is the y value for group 3, which appears to be too large, as is also suggested by the value of μ for this group.

VI. *The Mean Parallax.*

The correction found in IV. to the values of hV obtained above will affect the mean parallax; for small values of hV the mean parallax will be too small. The mean of all the groups is ″·0026, but if we omit the three small values of hV (see Table III.), for which the correction required is large, retaining only values over 0·30, for which the correction is quite small, we find $\pi = ″·00364$.

This value is in good agreement with the value $\pi = ″·00418$ found by the Astronomer Royal in the case of the Carrington stars, especially as the difference is in the same direction as the small outstanding error in hV.

This value would give a distribution in space very nearly the same as that given in the table in *Monthly Notices*, 1913 March, p. 342.†

Since these proper motions were determined from defective plates, it is probable that the limiting magnitude is not far from that of Carrington's Catalogue, viz. 10·3; if anything, it is probably a little fainter, which would also help to account for the slightly smaller value of the mean parallax found above.

* *M.N., R.A.S.*, 1911 May, p. 582.

† At the bottom of p. 341 there is a misprint, 1·872 being printed for 1·772 the value of $\sqrt{\pi}$, but the correct value has also been used in determining k.

VII. *Summary.*

Over 400 large proper motions—many of them of faint stars—determined at Oxford have been considered. The numbers of these proper motions between certain limits have been compared with the theoretical numbers, and the following facts emerge :—

(1) In the formula (1) above the value $\lambda = 1{\cdot}0$ suits the Oxford results well.
(2) The agreement between observation and theory is good, but the observed number of very large proper motions seems to be consistently in defect of the theoretical number.
(3) The mean value of a for this region is between $1{\cdot}5$ and $1{\cdot}7$.
(4) a clearly varies with galactic latitude. (a is, of course, the mean value of the cross proper motions, but it has been treated as a constant to be determined independently of this fact.)
(5) The mean parallax of these stars is slightly greater than ·00364, a value in good agreement with that found by the Astronomer Royal in the case of the Carrington stars.

Nizamiah Observatory, Hyderabad:
1915 *December* 15.

Ninth Note on the Number of Faint Stars with Large Proper Motions. By R. J. Pocock, B.A., B.Sc.

[*Introductory Note by Professor H. H. Turner.*—A word of explanation may be given why the following proper motions were determined at Hyderabad and not at Oxford. Some months ago Mr. Pocock feared that the Hyderabad Observatory might run short of plates to expose, owing to various difficulties in getting them since the outbreak of war. At his request, therefore, we sent him from Oxford a number of plates already exposed, being the repetitions, after an interval of at least ten years, of regions printed in the Astrographic Catalogue. He wished his staff to have something to measure if his fears were realised. Unfortunately they were: a long series of minor delays culminated in the loss of twenty-four dozen plates in the *Persia*. This loss was not realised for a week or two: thereupon the makers promptly set about replacing it, and supplies have probably reached Hyderabad by this time.* But there was an unexpectedly long period during which the Oxford plates were very welcome. On receipt of the measures of the new plates, Mr. Bellamy has care-

* Six dozen plates arrived at the Nizamiah Observatory on February 29, and satisfactory exposures were made the same night.

fully remeasured all cases of suspected P.M., cutting out a number where the revised differences fell short of the adopted limit. It was judged advisable to reject one whole plate which had diffused réseau lines.]

The Proper Motions given below have been determined by similar methods to those in the first eight Notes of the series.

In all, six regions have been compared: in three cases both plates were measured at the Nizamiah Observatory, Hyderabad, in each of the other three cases one plate was measured at Hyderabad and compared with the printed measures in the Oxford volumes. The plates were all taken at Oxford, but the comparisons were carried out at Hyderabad.

The total number of stars compared is 1065. The regions are given in Table I.

TABLE I.

List of Regions Compared.

Region.	Decl.	R.A.	Interval in Years.	No. of Stars.	No. of P.Ms.
	°	h m			
I.	+26	20 32	12·143	360	1
II.	+27	4 36	12·184	58	1
III.	+27	21 0	15·013	145	6
IV.	+27	21 32	16·082	243.	9
V.	+29	3 22	12·250	115	2
VI.	+29	12 22	15·118	144	7

The Proper Motions determined are given in Table II.; only cases in which the difference amounted to at least 1″·5 in one co-ordinate are included. The photo-mag. is in every case determined from the measured diameters by means of the formulæ given in the Oxford Catalogue. The Oxford numbers with an * denote bright stars contained in the Cambridge Catalogue; there are 12 out of 26.

TABLE II.

Large Proper Motions.

Ref. No.	Region.	Oxford No.	ϕ mag.	Δx. (unit 0″·3).	Δy.	Centennial P.M. in Arc.
	h m	°				″
1	21 32	+27 63364	10·3	+27	0	50·49
2	21 32	27 63114*	7·4	+17	− 2	31·98
3	4 36	27 9958*	8·7	+ 4	−10	26·57
4	20 32	26 66463*	10·2	− 8	− 6	24·81
5	21 0	27 61119*	8·1	− 8	− 9	24·09
6	3 22	29 8161	10·6	− 5	− 7	22·20
7	21 0	27 61082	9·7	+10	+ 1	20·10
8	21 0	27 61040	11·3	− 8	− 6	20·00

TABLE II.—*Large Proper Motions*—continued.

Ref. No.	Region.	Oxford No.	φ mag.	Δx. (unit 0″·3).	Δy. (unit 0″·3).	Centennial P.M. in Arc.
	h m	°				″
9	12 22	29 32604*	7·5	− 3	+ 8	16·90
10	21 32	27 63501	9·7	+ 7	+ 5	16·08
11	21 32	27 63015	10·4	− 2	− 8	15·30
12	3 22	29 8216	10·5	+ 6	− 1	14·90
13	12 22	29 32712*	10·6	+ 2	− 7	14·41
14	12 22	29 32635	10·3	− 7	+ 1	14·00
15	21 32	27 63535*	9·6	+ 6	+ 4	13·46
16	21 32	27 63208*	8·2	− 7	− 1	13·28
17	21 32	27 63486*	5·8	+ 6	+ 3	12·53
18	12 22	29 32671*	5·2	− 6	+ 2	12·51
19	21 32	27 63228*	8·9	+ 4	+ 5	11·97
20	21 32	27 63306	10·9	+ 6	+ 2	11·78
21	21 0	27 61121	11·1	− 5	− 3	11·67
22	12 22	29 32740	9·6	− 5	+ 3	11·54
23	12 22	29 32761	10·6	− 3	− 5	11·54
24	12 22	29 32772	8·4	+ 3	− 5	11·54
25	21 0	27 61166*	8·4	+ 5	− 1	10·20
26	21 0	27 61014	9·3	0	− 5	10·00

Nos. 2, 4, 18 (γ Comæ Beren.) are known Proper Motion stars (see Oxford Astrographic Catalogue, vol. viii. List D).

Nos. 5, 15, 17 are in List B, Oxford Catalogue, vol. viii., as stars which may have Proper Motion but need confirmation.

Nos. 1, 3 have appeared in previous papers of this series (Notes 1 and 2 respectively) centennial P.Ms. 42″·2 and 25″·0.

Two stars, Oxford +26° 67823, +27° 9751, for which Proper Motions have been given in Notes 2, 3 of this series, occur on plates used in the present investigation, but the Proper Motion is not confirmed; in the case of No. 67823, an error has now been found in the previous figures, which reduces the differences below the ·005 limit. Hence this star should be struck out of the list on p. 493 of *M.N.*, lxix. For +27° 9751 the former differences are found on scrutiny to be slightly too large; when corrected they would also fall below the adopted limit, though a Proper Motion remains of approximately the character printed.

Nizamiah Observatory,
Hyderabad:
1915 *November* 24.

Notes on the General Solution of Hill's Equation. By E. Lindsay Ince, M.A., B.Sc.(Edin.), Carnegie Fellow.

(*Communicated by Prof. H. F. Baker, F.R.S.*)

Contents of Paper.—In § 1 the solution of Hill's equation appropriate to the case of Θ_0 nearly equal to unity is taken, and the variation of Θ_0 and μ as the auxiliary parameter σ varies is discussed numerically, special regard being taken as to the stability or instability of the solution. In § 2 the solution applicable to the case of Θ_0 nearly equal to 4 is obtained and discussed in a similar manner, and reference is incidentally made to the cases of Θ_0 nearly equal to 9, 16 . . . . Finally, in § 3, the solution which diverges in the vicinity of $\Theta_0 = 1, 4, 9, \ldots$ is taken, and its relation to the other solutions is demonstrated.

§ 1. In a paper "On a General Solution of Hill's Equation," which I communicated last year to this Society,* I showed that Hill's equation

$$\frac{d^2y}{dz^2} + (\Theta_0 + 2\Theta_1 \cos 2z + 2\Theta_2 \cos 4z + \ldots)y = 0 \quad . \quad (1)$$

was satisfied by a function $e^{\mu z}u(z)$, where $u(z)$ was a purely periodic function of z of the form

$$\sin(z-\sigma) + a_3 \cos(3z-\sigma) + b_3 \sin(3z-\sigma) + a_5 \cos(5z-\sigma) + b_5 \sin(5z-\sigma) + \ldots,$$

and μ appeared as a function of σ and of the parameters Θ_1, Θ_2, etc. (but not Θ_0), of the differential equation. The new parameter σ is not arbitrary, for the quantity Θ_0 has to satisfy a certain relation involving σ, so that if Θ_0 is fixed, σ is determined also. By this means it was possible to evaluate μ by first determining σ from the known constants of the equation, and then, knowing σ, to compute the numerical value of μ.

This method of attacking the problem appears, from a theoretical point of view, so artificial that it has been suggested to me that I should investigate its real significance and so explain its apparent artificiality. These notes are therefore devoted to an inquiry into the properties of the new parameter σ, and incidentally to the question of the stability or instability of the solution corresponding to any given value of the parameter Θ_0, when the constants Θ_1, Θ_2, Θ_3, etc., have definite values assigned to them.†

For the purposes of this paper, Hill's values of the constants Θ_1, Θ_2, Θ_3, etc., will be adopted, but Θ_0 will, of course, be left quite arbitrary. As the series for Θ_0 and μ are unsuited for accurate numerical work when large imaginary or complex values of σ are inserted, an accuracy of only four or five places of

* *M.N.*, lxxv. p. 436, March 1915.

† For a somewhat similar investigation applied to Mathieu's equation, *vide* A. W. Young, *Proc. Edin. Math. Soc.*, xxxii. (1913–14), p. 81.

decimals will be aimed at. This degree of accuracy will, however, be adequate to illustrate the phenomena with which this paper professes to deal. The series for Θ_0 and μ obtained in my previous paper are as follows:—

$$\Theta_0 = 1 + \Theta_1 \cos 2\sigma + (-\tfrac{1}{4} + \tfrac{1}{8}\cos 4\sigma)\Theta_1^2 + \tfrac{1}{4}\Theta_1\Theta_2\cos 2\sigma - \tfrac{1}{64}\Theta_1^3\cos 2\sigma + (-\tfrac{11}{192} + \tfrac{5}{64}\cos 4\sigma)\Theta_1^2\Theta_2 + (\tfrac{1}{48} - \tfrac{11}{512}\cos 4\sigma)\Theta_1^4 + \ldots \quad (2)$$

$$\mu = \tfrac{1}{2}\Theta_1 \sin 2\sigma + \tfrac{1}{8}\Theta_1\Theta_2 \sin 2\sigma - \tfrac{3}{128}\Theta_1^3 \sin 2\sigma + \tfrac{1}{128}\Theta_1^2\Theta_2 \sin 4\sigma - \tfrac{3}{1024}\Theta_1^4 \sin 4\sigma + \ldots \quad (3)$$

which become, when Hill's values of the constants are inserted,

$$\Theta_0 = \cdot 99918\ 7 - \cdot 05704\ 66 \cos 2\sigma + \cdot 00040\ 662 \cos 4\sigma + \ldots \quad (4)$$

$$\mu = \cdot 02852\ 04 \sin 2\sigma + \cdot 00000\ 002 \sin 4\sigma - \ldots * \quad (5)$$

In the previous paper Hill's value of the quantity Θ_0 was assumed, and the corresponding value of σ deduced, but in the present notes Θ_0 will be regarded as a dependent variable whose behaviour, as σ varies, is to be investigated. For the present we see that if Θ_0 is real, then $\cos 2\sigma$ is real, and consequently σ is either (i.) purely real, or (ii.) purely imaginary, or (iii.) (as in Hill's case) a complex quantity of the form $\frac{\pi}{2}+$ a pure imaginary. In the first case μ also is purely real, but in the two other cases μ is a pure imaginary.

We will first consider the case when σ is purely real, in order that we may find out the limits within which Θ_0 lies, and also the extent of the variation of μ. For this purpose we may construct the table.

TABLE Ia.

σ.	Θ_0.	μ.
0	·94255	0
$\frac{\pi}{12}$	·94999	·01426
$\frac{\pi}{6}$	·97046	·02470
$\frac{\pi}{4}$	·99878	·02852
$\frac{\pi}{3}$	1·02791	·02470
$\frac{5\pi}{12}$	1·04879	·01426
$\frac{\pi}{2}$	1·05664	0
$\frac{7\pi}{12}$	1·04879	·01426
...	...	...

* For convenience the sign of μ has been changed, and in the following pages μ will always be regarded as positive.

We thus see that for values of Θ_0 lying between ·94255 . . . and 1·05664 . . . , μ is real and the corresponding solution unstable, its maximum instability being that introduced by the factor $e^{\cdot 02852 \ldots z}$ which arises when $\Theta_0 = \cdot 99878$ . . . .

When Θ_0 assumes either of the particular values ·94255 . . . or 1·05664 . . . μ becomes identically zero, and the corresponding solutions of Hill's equation are purely periodic with the period 2π. In general these purely periodic solutions are given by

$$\Theta_0 = 1 + \Theta_1 - \tfrac{1}{8}\Theta_1^2 + \tfrac{1}{4}\Theta_1\Theta_2 - \tfrac{1}{64}\Theta_1^3 + \tfrac{1}{48}\Theta_1^2\Theta_2 - \tfrac{1}{1536}\Theta_1^4 + \ldots \quad (6)$$

corresponding to $\sigma = 0$, and

$$\Theta_0 = 1 - \Theta_1 - \tfrac{1}{8}\Theta_1^2 - \tfrac{1}{4}\Theta_1\Theta_2 + \tfrac{1}{64}\Theta_1^3 + \tfrac{1}{48}\Theta_1^2\Theta_2 - \tfrac{1}{1536}\Theta_1^4 + \ldots \quad (7)$$

corresponding to $\sigma = \frac{\pi}{2}$.*

These two series define the unstable region which surrounds the value $\Theta_0 = 1$, and we see that the smaller the constants Θ_1, Θ_2 . . . , the more contracted is the region, and ultimately, when these constants vanish, no instability remains in the neighbourhood of $\Theta_0 = 1$.

The point of maximum instability in this region is given by $\sigma = \frac{\pi}{4}$, and so corresponds to

$$\Theta_0 = 1 - \tfrac{3}{8}\Theta_1^2 - \tfrac{13}{96}\Theta_1^2\Theta_2 + \tfrac{655}{15360}\Theta_1^4 + \ldots \quad (8)$$

the amount of this maximum instability being given by the exponent factor

$$\mu = \tfrac{1}{2}\Theta_1 + \tfrac{1}{8}\Theta_1\Theta_2 - \tfrac{3}{128}\Theta_1^3 + \ldots \quad (9)$$

Turning now to the imaginary and complex values of σ, we may draw up a similar table, exhibiting side by side the values of Θ_0 and μ, corresponding to values of σ of the form $i\rho$ and $\frac{\pi}{2} + i\rho$ respectively, where ρ is a purely real quantity and $i \equiv \sqrt{-1}$.

As has already been remarked, in these cases Θ_0 is purely real and μ is purely imaginary. We will first of all consider the last three columns of the following table, and observe that between the values $\frac{\pi}{2} + 2{\cdot}1i$ and $\frac{\pi}{2} + 2{\cdot}2i$ of σ, μ takes on the value i and the solution again becomes purely periodic, with a period π. Corresponding to this the quantity Θ_0 assumes a value very nearly equal to 4. But it will be shown in the next section that near $\Theta_0 = 4$ there are two such purely periodic solutions of the equation, viz. those corresponding to $\Theta_0 = 4{\cdot}00011$ . . . and $\Theta_0 = 4{\cdot}00091$ . . . respectively, and that to all values of Θ_0 intermediate to these

* These correspond to the cases in which Mathieu's equation gives rise to the elliptic cylinder functions $se_1(z)$ and $ce_1(z)$ respectively.

TABLE Ib.

σ.	Θ_0.	μ.	σ.	Θ_0.	μ.
0	·9426	0	$\frac{\pi}{2}$	1·0566	0
$\cdot 5i$	·9106	$\cdot 0334i$	$\frac{\pi}{2}+\cdot 5i$	1·0908	$\cdot 0334i$
$1\cdot 0i$	·7957	$\cdot 1034i$	$\frac{\pi}{2}+1\cdot 0i$	1·2249	$\cdot 1034i$
$1\cdot 5i$	·5097	$\cdot 2857i$	$\frac{\pi}{2}+1\cdot 5i$	1·6527	$\cdot 2857i$
$1\cdot 75i$	·2767	$\cdot 4718i$	$\frac{\pi}{2}+1\cdot 75i$	2·1675	$\cdot 4718i$
$2\cdot 0i$	·0474	$\cdot 7783i$	$\frac{\pi}{2}+2\cdot 0i$	3·1631	$\cdot 7783i$
$2\cdot 1i$	·0006	$\cdot 9508i$	$\frac{\pi}{2}+2\cdot 1i$	3·8054	$\cdot 9506i$
$2\cdot 2i$	·0239	$1\cdot 1615i$	$\frac{\pi}{2}+2\cdot 2i$	4·6705	$1\cdot 1610i$
$2\cdot 3i$	·1721	$1\cdot 4191i$	$\frac{\pi}{2}+2\cdot 3i$	5·8470	$1\cdot 4181i$
$2\cdot 4i$	·5318	$1\cdot 7334i$	$\frac{\pi}{2}+2\cdot 4i$	7·4623	$1\cdot 7313i$
$2\cdot 5i$	1·2441	$2\cdot 1189i$	$\frac{\pi}{2}+2\cdot 5i$	9·7102	$2\cdot 1136i$

correspond real values of μ, and, in consequence, unstable solutions. But no real values of μ could possibly appear in our table (Ib.), since every term in the series for μ is now a pure imaginary. The only explanation of this peculiarity which readily suggests itself is that the expansion of Θ_0 as a series in σ is discontinuous near the value 4; that is to say, as σ varies from $\frac{\pi}{2}$ to $\frac{\pi}{2}+2\cdot 125\ldots i$,* Θ_0 varies continuously from 1·0566 to 4·0001 . . . , and then, as σ is further increased, suddenly jumps to 4·0009 . . . and then continues for a time to increase steadily. Now as σ continues further to increase we see from the table that μ soon approaches a value $2i$, which again leads to a purely periodic solution corresponding to a value of Θ_0 very nearly equal to 9. But it will be shown in the next section that near $\Theta_0=9$ there are two purely periodic solutions, corresponding respectively to $\Theta_0=9\cdot 00019\ 689\ldots$ and $\Theta_0=9\cdot 00021\ 015\ldots$. As before, the intermediate values of Θ_0 give rise to an unstable solution, and thus the series for Θ_0 in terms of σ would be expected to be discontinuous near $\Theta_0=9$. Likewise two values of Θ_0 near·16 can be found which give rise to purely periodic solutions, and here again the intermediate values give rise to instability. Thus in this region also the series for Θ_0 would naturally be expected to exhibit a discontinuity. Unfortunately, however, the convergence of that series

* This value is obtained by inverse interpolation in the table, and corresponds to $\mu=i$.

rapidly diminishes together with its applicability to numerical work, and for this reason the table has been stopped before any such discontinuity could be reached. Similarly, regions of instability can be shown to exist near $\Theta_0 = 25, 36, 49 \ldots$, and there the series for Θ_0, if convergent, would most probably become discontinuous.

Now let us examine the values of Θ_0 and μ, which correspond to a purely imaginary σ (Table Ib.). As σ increases numerically so also does μ, but Θ_0 decreases at first until when $\sigma = 2{\cdot}12 \ldots i$, $\mu = i$, and Θ_0 has a small negative value. Thus we have a purely periodic solution.* But it will be shown in § 3 that no other periodic solutions exist for values of Θ_0 less than the present one ($\Theta_0 = -{\cdot}00163 \ldots$), and that for all such values of Θ_0 the solution is unstable. Thus the series in σ does not become discontinuous at this point, but has a minimum after which Θ_0 begins steadily to increase until between $\sigma = 2{\cdot}4i$ and $\sigma = 2{\cdot}5i$, μ attains the value i and Θ_0 (which is then nearly unity) has a discontinuity. Thereafter Θ_0 continues steadily to increase, and the range of values of Θ_0 and μ contained in the last three columns of the table are reproduced, so that the first three columns give us no further information than that already obtained. Thus we see that the series for Θ_0 in terms of σ would give the complete set of possible values of Θ_0 that give rise to stability and omit all those that produce instability. There seems at present to be no means of definitely proving the existence of discontinuities in the series considered, but there seems to be little doubt that they actually do exist.

§ 2. At the conclusion of my paper "On a General Solution of Hill's Equation" I remarked that though the solution I there gave was only of value in the case when Θ_0 was nearly unity, it was equally possible to obtain solutions appropriate to the cases when Θ_0 was approximately 4, 9, etc. I now propose to deal with such cases, and to link them up with the former case.

As before, we assume a solution of the form

$$y = e^{\mu z} u(z)$$

where

$$\mu = p_1(\sigma)\Theta_1 + p_2(\sigma)\Theta_2 + \ldots + q_1(\sigma)\Theta_1^2 + q_2(\sigma)\Theta_2^2 + \ldots + q_{12}(\sigma)\Theta_1\Theta_2 + \ldots + r_1(\sigma)\Theta_1^3 + \ldots,$$

but in this case

$$u(z) = \sin(2z - \sigma) + A_1(z, \sigma)\Theta_1 + A_2(z, \sigma)\Theta_2 + \ldots + B_1(z, \sigma)\Theta_1^2 + B_2(z, \sigma)\Theta_2^2 + B_{12}(z, \sigma)\Theta_1\Theta_2 + \ldots C_1(z, \sigma)\Theta_1^3 + \ldots,$$

the functions A, B, C, . . . being conditioned to contain no terms in either $\sin(2z - \sigma)$ or $\cos(2z - \sigma)$.

The relation between Θ_0 and σ is in this case

$$\Theta_0 = 4 + \lambda_1(\sigma)\Theta_1 + \lambda_2(\sigma)\Theta_2 + \ldots + \mu_1(\sigma)\Theta_1^2 + \mu_2(\sigma)\Theta_2^2 + \ldots + \mu_{12}(\sigma)\Theta_1\Theta_2 + \ldots + \nu_1(\sigma)\Theta_1^3 + \ldots,$$

* In fact, that one which corresponds to the Mathieu function $ce_0(z)$.

giving us a solution applicable in particular to the case in which Θ_0 does not differ greatly from the value 4.

Proceeding exactly as before, we find that the terms in the general solution are as follows. (Only those terms whose order is the eighth or less than the eighth are investigated, as they will be sufficient for all present purposes.)

Terms in Θ_1—

$$p_1 = 0.$$
$$\lambda_1 = 0.$$
$$A_1 = \tfrac{1}{4}\sin\sigma + \tfrac{1}{12}\sin(4z-\sigma).$$

Terms in Θ_2—

$$p_2 = \tfrac{1}{4}\sin 2\sigma.$$
$$\lambda_2 = \cos 2\sigma.$$
$$A_2 = \tfrac{1}{32}\sin(6z-\sigma).$$

Terms in Θ_r, *where* $r>2$—

$$p_r = 0.$$
$$\lambda_r = 0.$$
$$A_r = \frac{1}{4r(r+2)}\sin\{(2r+2)z-\sigma\} - \frac{1}{4r(r-2)}\sin\{(2r-2)z+\sigma\}.$$

Terms in Θ_1^2—

$$q_1 = -\tfrac{1}{16}\sin 2\sigma.$$
$$\mu_1 = \tfrac{1}{6} - \tfrac{1}{4}\cos 2\sigma.$$
$$B_1 = \tfrac{1}{384}\sin(6z-\sigma).$$

Terms in Θ_2^2—

$$q_2 = 0.$$
$$\mu_2 = -\tfrac{3}{64} + \tfrac{1}{32}\cos 4\sigma.$$
$$B_2 = \tfrac{1}{6144}\sin(10z-\sigma) + \tfrac{3}{10240}\sin 2\sigma\cos(6z \quad \sigma) + \tfrac{1}{10240}\cos 2\sigma\sin(6z-\sigma).$$

Terms in Θ_r^2, *where* $r>2$—

$$q_r = 0.$$
$$\mu_r = -\frac{1}{2(r^2-4)}.$$

Terms in $\Theta_1\Theta_2$—

$$q_{12} = 0.$$
$$\mu_{12} = 0.$$
$$B_{12} = \tfrac{11}{57600}\sin(8z-\sigma) + \tfrac{5}{1440}\sin 2\sigma\cos(4z-\sigma) + (\tfrac{1}{384}\cos 2\sigma - \tfrac{7}{3840})\sin(4z-\sigma) - \tfrac{1}{320}\sin 3\sigma + \tfrac{5}{96}\sin\sigma.$$

Terms in $\Theta_1\Theta_3$—

$$q_{13} = \tfrac{1}{24} \sin 2\sigma.$$
$$\mu_{13} = \tfrac{1}{6} \cos 2\sigma.$$

Terms in $\Theta_1\Theta_r$, *where* $r > 3$—

$$q_{1r} = 0.$$
$$\mu_{1r} = 0.$$

Terms in $\Theta_1{}^3$—

$$r_1 = 0.$$
$$\nu_1 = 0.$$
$$C_1 = \tfrac{1}{2304} \sin(8z - \sigma) - \tfrac{1}{288} \sin 2\sigma \cos(4z - \sigma) + (\tfrac{19}{13824} - \tfrac{1}{576} \cos 2\sigma) \sin(4z - \sigma) + \tfrac{1}{128} \sin 3\sigma - \tfrac{7}{384} \sin \sigma.$$

Terms in $\Theta_1{}^2\Theta_2$—

$$r_{12} = -\tfrac{1}{72} \sin 2\sigma + \tfrac{1}{128} \sin 4\sigma.$$
$$\nu_{12} = \tfrac{1}{12} - \tfrac{1}{9} \cos 2\sigma + \tfrac{1}{64} \cos 4\sigma.$$

Terms in $\Theta_1{}^4$—

$$s_1 = \tfrac{1}{288} \sin 2\sigma - \tfrac{1}{512} \sin 4\sigma.$$
$$\varpi_1 = -\tfrac{1}{6}\tfrac{49}{912} + \tfrac{1}{36} \cos 2\sigma - \tfrac{3}{512} \cos 4\sigma.$$
$$D_1 = \tfrac{1}{221184} \sin(10z - \sigma) - \tfrac{25}{1474560} \sin 2\sigma \cos(6z - \sigma) + (\tfrac{1}{172800} - \tfrac{11}{1474560} \cos 2\sigma) \sin(6z - \sigma).$$

Terms in $\Theta_1{}^5$—

$$t_1 = 0.$$
$$\rho_1 = 0.$$

We thus obtain, as the required expansions of Θ_0 and μ, the following:—

$$\Theta_0 = 4 + \Theta_2 \cos 2\sigma + (\tfrac{1}{6} - \tfrac{1}{4} \cos 2\sigma)\Theta_1{}^2 + (-\tfrac{3}{64} + \tfrac{1}{32} \cos 4\sigma)\Theta_2{}^2 + (\tfrac{1}{12} - \tfrac{1}{9} \cos 2\sigma + \tfrac{1}{64} \cos 4\sigma)\Theta_1{}^2\Theta_2 + (-\tfrac{1}{6}\tfrac{49}{912} + \tfrac{1}{36} \cos 2\sigma - \tfrac{3}{512} \cos 4\sigma)\Theta_1{}^4 + \ldots \quad (10)$$

$$\mu = \tfrac{1}{4}\Theta_2 \sin 2\sigma - \tfrac{1}{16}\Theta_1{}^2 \sin 2\sigma + (-\tfrac{1}{72} \sin 2\sigma + \tfrac{1}{128} \sin 4\sigma)\Theta_1{}^2\Theta_2 + (\tfrac{1}{288} \sin 2\sigma - \tfrac{1}{512} \sin 4\sigma)\Theta_1{}^4 + \ldots \quad (11)$$

As before, when σ is real μ also is real, and consequently the solution is unstable, and when σ is imaginary μ is imaginary, and the solution is stable. When σ assumes either of the values 0 or $\frac{\pi}{2}$, μ vanishes, and the solution becomes purely periodic with

a period π. The corresponding values of Θ_0 in these cases are respectively given by the relations

$$\Theta_0 = 4 + \Theta_2 - \tfrac{1}{12}\Theta_1^2 - \tfrac{1}{64}\Theta_2^2 - \tfrac{7}{576}\Theta_1^2\Theta_2 + \tfrac{5}{13824}\Theta_1^4 + \ldots \quad (12)$$

and

$$\Theta_0 = 4 - \Theta_2 + \tfrac{5}{12}\Theta_1^2 - \tfrac{1}{64}\Theta_2^2 + \tfrac{121}{576}\Theta_1^2\Theta_2 - \tfrac{763}{13824}\Theta_1^4 + \ldots \quad (13)^*$$

Adopting now Hill's values of the constants Θ_1, Θ_2 . . ., the expansions of Θ_0 and μ become

$$\Theta_0 = 4{\cdot}00054 - {\cdot}00043 \cos 2\sigma + \ldots \quad (14)$$

and

$$\mu = {\cdot}00010\ 7 \sin 2\sigma + \ldots \quad (15)$$

Thus when σ is purely real, Θ_0 and μ assume the values given in the following table.

TABLE IIa.

σ.	Θ_0.	μ.
0	4·00011	0
$\frac{\pi}{12}$	4·00017	·00005
$\frac{\pi}{6}$	4·00033	·00009
$\frac{\pi}{4}$	4·00054	·00011
$\frac{\pi}{3}$	4·00076	·00009
$\frac{5\pi}{12}$	4·00091	·00005
$\frac{\pi}{2}$	4·00097	0
$\frac{7\pi}{12}$	4·00091	·00005
...	...	...

All values of Θ_0 lying within the interval between $\Theta_0 = 4{\cdot}00011$ and $\Theta_0 = 4{\cdot}00097$ give rise to real values of μ, and consequently to an unstable solution, the maximum instability being that represented by the factor $e^{\cdot 00010 \cdots z}$. It will at once be noticed that the interval within which instability would occur is considerably smaller than in the case in which Θ_0 is nearly unity, and further, that the secular factor is enormously less in this case than in the last. If we turn for a moment to the case in which Θ_0 is nearly 9, we find that we may obtain expansions of the form

$$\Theta_0 = 9 + \Theta_3 \cos 2\sigma + \tfrac{1}{16}\Theta_1^2 + \tfrac{1}{10}\Theta_2^2 - \tfrac{1}{4}\Theta_1\Theta_2 \cos 2\sigma + \tfrac{1}{64}\Theta_1^3 \cos 2\sigma + \tfrac{13}{20480}\Theta_1^4 + \ldots \quad (16)$$

* In the case of Mathieu's equation these would correspond to the unctions $se_2(z)$ and $ce_2(z)$ respectively.

and $$\mu = \tfrac{1}{6}\Theta_3 \sin 2\sigma - \tfrac{1}{24}\Theta_1\Theta_2 \sin 2\sigma + \tfrac{1}{384}\Theta_1{}^3 \sin 2\sigma + \ldots \quad (17)$$

which, with Hill's numbers, become

$$\Theta_0 = 9{\cdot}00020\ 352 - {\cdot}00000\ 663 \cos 2\sigma + \ldots \quad (18)$$

and

$$\mu = {\cdot}00000\ 110 \sin 2\sigma + \ldots \quad (19)$$

Thus in this case the interval during which μ is real is extremely short, and the maximum real value of μ itself is only ·000001, and hence the region of instability near $\Theta_0 = 9$ is very limited, and the amount of instability is at most relatively minute. This applies generally to the higher expansions of Θ_0 and μ, that is to say, to those applicable in the neighbourhood of $\Theta_0 = 16, 25, 36 \ldots$. So that, as Θ_0 increases in magnitude, the possibility of its being associated with a markedly unstable solution becomes less and less.

Returning now to equations (12) and (13), and tabulating values of Θ_0 and μ corresponding to the two possible sets of non-real values of σ, we obtain the following:—

TABLE IIB.

σ.	Θ_0.	μ.	σ.	Θ_0.	μ.
0	4·00011	0	$\frac{\pi}{2}$	4·00097	0
$1{\cdot}0i$	3·99849	$·00039i$	$\frac{\pi}{2} + 1{\cdot}0i$	4·00173	$·00039i$
$2{\cdot}0i$	5·9888	$·0029i$	$\frac{\pi}{2} + 2{\cdot}0i$	4·0123	$·0029i$
$3{\cdot}0i$	3·914	$·022i$	$\frac{\pi}{2} + 3{\cdot}0i$	4·087	$·022i$
$4{\cdot}0i$	3·35	$·16i$	$\frac{\pi}{2} + 4{\cdot}0i$	4·65	$·16i$
...	...	...	...	...	...

The tendency in this case seems to be as in the previous one, but the series are not sufficiently convergent to show the phenomena in their full detail.

§ 3. The method of dealing with Hill's equation which has concerned us so far is suited to the cases in which Θ_0 is nearly 1, 4, 9, . . . but not to the case of Θ_0 approximately zero. In order to complete the discussion we will obtain a solution appropriate to this case. To do so we assume a solution

$$y = e^{cz} u$$

where c is a constant and

$$u = 1 + A_1(z)\Theta_1 + A_2(z)\Theta_2 + \ldots + B_1(z)\Theta_1{}^2 + B_2(z)\Theta_2{}^2 + \ldots + B_{12}(z)\Theta_1\Theta_2 + \ldots + C_1(z)\Theta_1{}^3 + \ldots$$

Θ_0 is now to be determined by the relation

$$\Theta_0 = a_0 + a_1\Theta_1 + a_2\Theta_2 + \ldots + b_1\Theta_1^2 + b_2\Theta_2^2 + \ldots + b_{12}\Theta_1\Theta_2 + \ldots + c_1\Theta_1^3 + \ldots$$

On substituting these expressions in Hill's equation and equating to zero the terms of the result which contain no Θ, we have

$$a_0 = -c^2.$$

We now proceed as usual, and obtain the following terms in the expansions of Θ_0 and u:—

Terms in Θ_n, *where* $n = 1, 2, 3 \ldots$—

$$a_n = 0.$$

$$A_n = \frac{1}{2(c^2+n^2)}\cos 2nz - \frac{c}{2n(c^2+n^2)}\sin 2nz.$$

Terms in Θ_n^2, *where* $n = 1, 2, 3$—

$$b_n = -\frac{1}{2(c^2+n^2)}.$$

$$B_n = \frac{2n^2-c^2}{16n^2(c^2+n^2)(c^2+4n^2)}\cos 4nz - \frac{3c}{16n(c^2+n^2)(c^2+4n^2)}\sin 4nz.$$

Terms in $\Theta_r\Theta_s$, *where* $r, s = 1, 2, 3 \ldots$ *and* $r \neq s$—

$$b_{rs} = 0.$$

Terms in Θ_n^3, *where* $n = 1, 2, 3 \ldots$—

$$c_n = 0.$$

Term in Θ_1^4—

$$d_1 = \frac{1-c^2}{16(1+c^2)^3} - \frac{1-2c^2}{32(1+c^2)^2(4+c^2)}.$$

Thus the following expression for Θ_0 in terms of Θ_1, Θ_2, and c is obtained, and will be sufficient for our purpose, viz.

$$\Theta_0 = -c^2 - \frac{1}{2(1+c^2)}\Theta_1^2 - \frac{1}{2(4+c^2)}\Theta_2^2 + \left[\frac{1-c^2}{16(1+c^2)^3} - \frac{(1-2c^2)}{32(1+c^2)^2(4+c^2)}\right]\Theta_1^4 + \ldots \quad (20)$$

which, with Hill's numbers, becomes

$$\Theta_0 = -c^2 - \frac{\cdot 00162\ 70100}{1+c^2} - \frac{\cdot 00000\ 00734}{4+c^2} + \frac{7-5c^2}{(1+c^2)^3(4+c^2)}(\cdot 00000\ 03309) + \ldots \quad (21)$$

This expression is quite unsuited for dealing with the purely periodic solutions which occur at $c = i,\ 2i,\ 3i\ \ldots$, as these are precisely the values of c which render it divergent. It is, however, useful for dealing with those solutions which are intermediate between the periodic solutions, and also enables us to dispose of the whole range of negative values of Θ_0.

When c is zero we have a periodic solution given by

$$\Theta_0 = -\tfrac{1}{2}\Theta_1^2 - \tfrac{1}{8}\Theta_2^2 - \ldots + \tfrac{7}{128}\Theta_1^4 + \ldots \qquad (22)^*$$

In Hill's case the value of this expression is $-\cdot00162\ldots$. This is the periodic solution of lowest order, for, corresponding to all real values of c, equation (21) gives us negative values of Θ_0 extending uninterruptedly down to negative infinity. We thus conclude that to all values of Θ_0 less than $-\cdot00162\ldots$ correspond unstable solutions of Hill's equation, and the larger Θ_0 is numerically, the greater is the amount of instability.

We will now consider imaginary values of c, and to treat of them we construct the following table :—

TABLE III.

c.	Θ_0.	c.	Θ_0.
0	$-\cdot00163$		
$\cdot1i$	$+\cdot00836$	$1\cdot1i$	$1\cdot21758$
$\cdot2i$	$\cdot03831$	$1\cdot2i$	$1\cdot44364$
$\cdot3i$	$\cdot08821$	$1\cdot3i$	$1\cdot69235$
$\cdot4i$	$\cdot15806$	$1\cdot4i$	$1\cdot96169$
$\cdot5i$	$\cdot24783$	$1\cdot5i$	$2\cdot25130$
$\cdot6i$	$\cdot35746$	$1\cdot6i$	$2\cdot56104$
$\cdot7i$	$\cdot48682$	$1\cdot7i$	$2\cdot89000$
$\cdot8i$	$\cdot63550$	$1\cdot8i$	$3\cdot24072$
$\cdot9i$	$\cdot80160$	$1\cdot9i$	$3\cdot61062$
...	...	...	...

This table may be continued indefinitely, giving us the values of Θ_0 which correspond to all values of c except those near the critical points $c = i,\ 2i,\ 3i\ \ldots$. In fact, the formulæ (20) and (21), though useless near these critical values, are much more useful than those of §§ 1, 2, for values of c remote from the critical values. On the other hand, the formulæ of §§ 1, 2 are particularly adapted to investigations near the appropriate critical values of c (or μ), and (unless a large number of terms in the series are used) relatively useless away from the particular critical values with which they are respectively associated. Too much weight must therefore not be given to the figures in Table IB., which was computed with series of only a limited number of terms, and which was constructed merely to show the tendency of the series in σ when σ was large and imaginary.

* This periodic solution corresponds to the Mathieu function $ce_0(z)$.

Reverting the series (20), we obtain (*cf.* Tisserand, *Méc. Cél.*, iii. p. 10)

$$c^2 = -\Theta_0 - \frac{\Theta_1^2}{2(1-\Theta_0)} - \frac{\Theta_2^2}{2(4-\Theta_0)} - \left[\frac{3-\Theta_0}{16(1-\Theta_0)^3} + \frac{1+2\Theta_0}{32(1-\Theta_0)^2(4-\Theta_0)}\right]\Theta_1^4 - \ldots \quad (23)$$

which, when the solution is stable, may be used for determining c when the constants Θ_0, Θ_1, Θ_2 . . . are given. Thus, using Hill's numbers, we have

$$c^2 = -1{\cdot}158844 + {\cdot}010246 + {\cdot}000289.$$
$$= -1{\cdot}148309.$$

The correct value of c^2 to six places of decimals is $-1{\cdot}148281$. This formula, in fact, does not give so good an approximation as do the formulæ involving σ when terms of the same order in Θ_1, Θ_2 . . . are retained in the former as in the latter, so that for problems such as Hill's, in which Θ_0 is near a critical value, the method of introducing the parameter σ is to be preferred.

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1916 *January* 25.

A Method for Calculating Longitude from Observations on the Moon and Moon-Culminating Stars. By F. J. Broadbent, B.Sc., A.M.Inst.C.E., Lecturer in Civil Engineering in the University of Bristol. (Plate 7.)

(*Communicated by S. Brodetsky.*)

Of all the absolute methods for determining longitude this is the simplest and best for a surveyor with a good transit theodolite. It is not the object of this paper to deal in detail with the methods of observation. These and other means of finding longitude are admirably dealt with in Close's *Topographical Surveying*. The principle depends on the fact that the moon's Right Ascension is constantly varying, and for any given Right Ascension the moon is on some particular meridian. The *Nautical Almanac* tabulates for each day of the year the R.A. of the moon's bright limb on the meridian of Greenwich and the meridian 180° or 12 hours from Greenwich, together with the time of passage of the semi-diameter over the meridian and the R.A. of stars (moon-culminating stars) whose declinations and R.A.'s do not differ much from that of the moon.

The observation consists in setting up the theodolite with its line of collimation on the meridian so that the passage over the central cross-hair of any object is its passage over the meridian. The time interval between the moon's bright limb touching this

cross-hair and the passage of a star gives a means for determining the R.A. of the moon's bright limb at the instant it was on the meridian, and hence of determining longitude. The method given in Close's *Topographical Surveying* for the reduction of the results is an application of the Theory of Finite Differences. A value of longitude is guessed as near to the true value as possible, and this value is used in the equation which is then used to determine the R.A. of the moon's bright limb corresponding with this longitude. In practice this means that at least two values must be guessed, and the final result obtained by interpolation. When the value of the R.A. found in this way agrees with that found by observation the problem is solved, and the longitude is the true one for the observations. The calculations are fairly laborious, and the author believes that the following alternative semi-graphical method has the advantage of simplicity of calculation.

Having obtained the R.A. of the moon's bright limb at the moment of observation, the procedure is as follows:—Plot a graph of about 4 or 5 points, using successive values of the R.A. of the moon's bright limb as abscissæ, and the sidereal time of the passage of the semi-diameter as ordinates. The central point should be that nearest the R.A. at the moment of observation. As the sidereal time of the passage of the semi-diameter varies very slowly, a very open scale may be used for this, so that ·01 second can be read on it. From this graph the sidereal time of the passage of the semi-diameter at the time the bright limb was on the meridian can be directly read. Using this reading, the *R.A. of the moon's centre* when on the meridian is determined.

The *Nautical Almanac* also tabulates the R.A. of the moon for each hour of the year, together with the variation in 10 minutes at each hour. Using this table, a second graph is plotted with the R.A. of the moon's centre as abscissæ and the variation in 10 minutes as ordinates, taking about four consecutive points, and arranging so that the R.A. at the moment of observation comes near the middle of the graph. The variation in 10 minutes alters very slowly over such a period, and may be plotted to a scale from which ·0002 second can be read. From this graph the ordinate midway between the nearest previous tabulated R.A. and the R.A. at the moment of the passage is taken as the mean for the intervening time (since the graph is practically a straight line), and using this variation the exact Greenwich Mean Time at the moment of the passage of the moon's centre is calculated. The G.M.T. is now reduced to sidereal time, and the difference between this and the R.A. of the moon's centre on crossing the meridian, *i.e.* local sidereal time, gives the longitude of the place. Should the moon pass through the "full" phase for the points plotted in the first graph, it is easy to find the time of passage of the same limb for all the points by correcting with twice the sidereal time of the semi-diameter crossing the meridian for those points where the moon has changed.

Two examples are added, one for longitude west and the other

for longitude east. In both cases the results were inserted in the usual equation given by Close, and gave Right Ascensions of the bright limb within ·03 and ·01 second of the observed results respectively. One of these calculations is shown in detail.

The defect of the method lies in observational difficulties. Since the moon's right ascension only varies about 2 seconds per minute, an error of timing of ·2 second makes a difference of 6 seconds of time in longitude.

Longitude by Observations on Moon and ϵ Geminorum.
1916 January 17.

Data.—Time interval from passage across meridian of moon's bright limb to passage of ϵ Geminorum was 30 minutes 3·8 seconds. Mean Time Units.

Calculations.

M.T. Intervals. m	s	S.T. Equivalents. m	s
30		30	4·93
	3·8		3·81
30	3·8	30	8·74

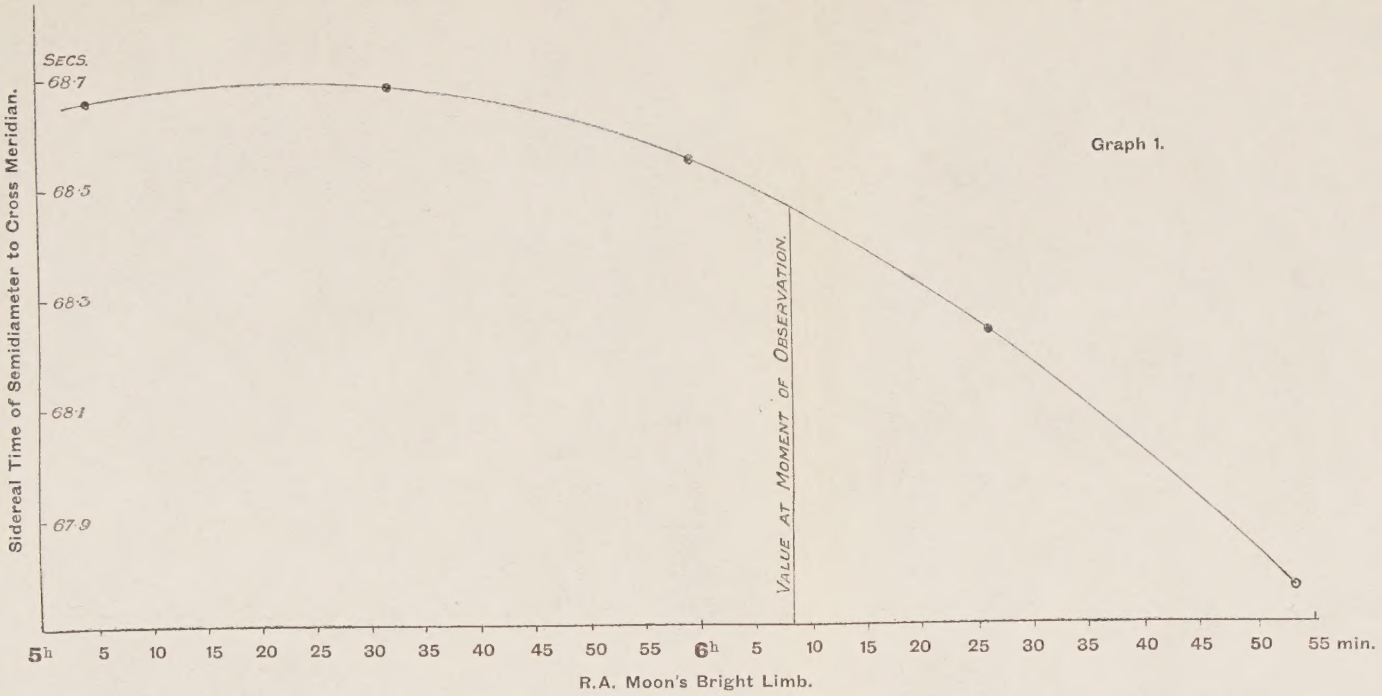
	h	m	s	
Right Ascension ϵ Geminorum . . .	6	38	48·59	
S.T. interval since moon's passage . .		30	8·74	
R.A. moon's bright limb on crossing observer's meridian	6	8	39·85	
Sidereal time for semi-diameter to cross meridian (Graph 1)		1	8·44	
R.A. moon's centre on crossing meridian .	6	9	48·29	= L.S.T.
R.A. at G.M.T. 14 hours	6	8	32·74	
Difference . .		1	15·55	= 75·55 seconds.

Mean variation in 10 minutes (Graph 2) = 21·9965 seconds.

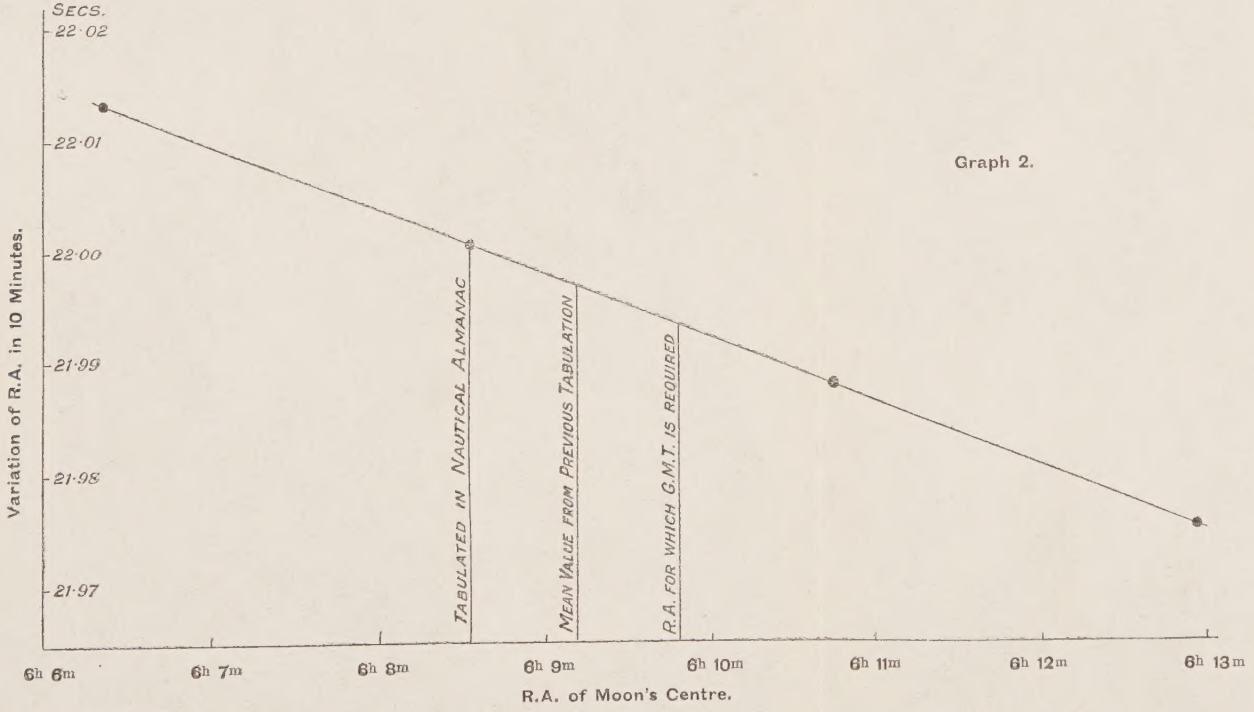
Time from 14 hours $= \frac{75{\cdot}55}{2{\cdot}19965} = 34{\cdot}3464$ minutes = 34 minutes 20·78 seconds.

G.M.T. at moment of moon's centre crossing meridian $14^h\ 34^m\ 20^s{\cdot}78$.

M.T. Intervals. h	m	s		S.T. Equivalents. h	m	s
14				14	2	17·99
	34				34	5·59
		20·78				20·84
14	34	20·78	S.T. interval from prev. G.M.N. .	14	36	44·42
			S.T. at prev. G.M.N. . .	19	42	21·12
				34	19	5·54
				24	0	0
			At same instant { G.S.T. . . .	10	19	5·54
			{ L.S.T. . . .	6	9	48·30
			Longitude west . .	4	9	17·24


SECS.
68·7
68·5
68·3
68·1
67·9
Sidereal Time of Semidiameter to Cross Meridian.
VALUE AT MOMENT OF OBSERVATION.
5h 5 10 15 20 25 30 35 40 45 50 55 6h 5 10 15 20 25 30 35 40 45 50 55 min.
R.A. Moon's Bright Limb.

Graph 1.



Graph 2.

Check of Foregoing Result.

R.A. Moon's Bright Limb. (From *Nautical Almanac.*)	1st Difference.	2nd Difference.	3rd Difference.	4th Difference.
h m s	m s	s	s	s
5 4 18·09	27 27·20	− ·92		
			− 8·21	
5 31 45·29	27 26·28	− 9·13 (C)		+ ·63 (E)
			− 7·58 (D)	
5 59 11·57 (A)	27 17·15 (B)	− 16·71		
6 26 28·72	27 0·44			
6 53 29·16				

The central value in the first column is the tabulated R.A. occurring first before the observation. The R.A.'s in the whole column are successive values taken directly from the *Nautical Almanac.*

The underlined values are the ones used in the calculation. If, taken in order, these be called A, B, C, D, and E, and if L is longitude west in hours, a value $n = \frac{L}{12}$ is used in the equation

$$\text{R.A.} = A + n \cdot B + \frac{n(n-1)}{1 \cdot 2} C + \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3} D + \frac{(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4} E$$

where R.A. is the Right Ascension of the moon's bright limb at the instant it crosses the meridian.

Care must be exercised in the use of the signs. The value of n becomes $\frac{12 - L}{12}$ for longitude east (see Close's *Topographical Surveying*).

In the case under consideration

$$L = 4^h\ 9^m\ 17{\cdot}24^s = 4{\cdot}15479^h$$

$$n = \frac{4{\cdot}15479}{12} = {\cdot}34623$$

$$\text{R.A.} = (5^h\ 59^m\ 11^s{\cdot}57) + {\cdot}34623(27^m\ 17^s{\cdot}15) + \frac{{\cdot}34623 \times {\cdot}65377}{2} \times 9^s{\cdot}13 + \frac{1{\cdot}34623 \times {\cdot}34623 \times {\cdot}65377}{6} \times 7^s{\cdot}58 + \frac{1{\cdot}346 \times {\cdot}346 \times {\cdot}6538 \times 1{\cdot}6538}{24} \times {}^s{\cdot}63$$

$$= (5^h\ 59^m\ 11^s{\cdot}57) + (9^m\ 26^s{\cdot}83) + (1^s{\cdot}03) + ({}^s{\cdot}38) + ({}^s{\cdot}01)$$
$$= 6^h\ 8^m\ 39^s{\cdot}82$$

The result obtained by direct observation was $6^h\ 8^m\ 39^s{\cdot}85$, so that it is well within the probable accuracy of the method.

In the second case the observation is for 1909 March 14, the star observed being λ Sagittarii.

The R.A. of the bright limb was $17^h\ 30^m\ 6^s{\cdot}98$ at the moment of crossing the meridian.

By the method given in this paper longitude was found to be $4^h\ 15^m\ 1^s{\cdot}05$ E., and using this value in the equation the R.A. of the bright limb worked out to $17^h\ 30^m\ 6^s{\cdot}99$.

Note.—There is a further source of error due to the present tabulation of the R.A.'s of the moon-culminating stars in the *Nautical Almanac*. In the current year these values are given to single seconds only. In the case of ϵ Geminorum the value used above was from the "apparent places." Had the value tabulated under Moon-Culminating Stars been used, there would have been an error of 11 seconds (time) of longitude in the result.